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Abstract

We analyze the effect of generalized first and second order stochastic dominance changes in a returns distribution on optimal financial portfolios with two risky and a risk free assets. We show that constant relative risk aversion plays an important role in explaining how the composition of the portfolios is affected. The results are interpreted in terms of two-fund separation.

Keywords: Stochastic dominance, first order stochastic dominance, second order stochastic dominance, relative risk aversion, financial portfolio, two-fund separation.

JEL classification numbers: D80, G11.

Résumé

Nous analysons les effets des dominances stochastiques des premier et second ordres des distributions de rendements sur la composition des portefeuilles financiers optimaux composés de deux actifs risqués et d'un actif sans risque. Nous montrons que la mesure d'aversion relative pour le risque joue un rôle important dans l'explication de la variation de la composition des portefeuilles. Les résultats sont interprétés dans un cadre d'une séparation à deux fonds.

Mots clés : Dominance stochastique, dominance stochastique de premier ordre, dominance stochastique de deuxième ordre, aversion relative au risque, portefeuille financier, séparation à deux fonds.

Classification JEL : D80, G11.

1 Introduction

Since the contribution of Rothschild and Stiglitz [1970, 1971] many models have been proposed to obtain intuitive results on the optimal behavior of risk averse individuals following both first and second-degree stochastic dominance changes in a returns distribution (FSD and SSD, respectively). For example, Meyer and Orminston [1985] proposed restrictions on distributions functions, Meyer and Orminston [1994] and Dionne and Gollier [1996] considered non independent risks and Eeckhoudt, Gollier and Schlesinger [1996] determined conditions for taking into account background changes in risk for independent risky assets (for a recent review of these results, see Eeckhoudt and Gollier, 2000). All these studies were limited to one decision variable, so they were not able to analyze the effects of generalized FSD and SSD on optimal financial portfolio with three assets or more.

The aim of this note is to propose such analysis for independent risky assets. An interpretation of our results in terms of mutual fund separation (Cass and Stiglitz, 1970) is proposed in the concluding remarks.

2 The model

We consider a risk averse agent who allocates his wealth (normalized to one) between one risk free asset (with return x_0) and two risky assets with returns $\tilde{x}_i$ for $i = 1, 2$. The portfolio share of risky asset $\tilde{x}_i$ is α_i. The agent's end of period wealth W is then equal to

$$W(\alpha_1, \alpha_2) = 1 + x_0 + \alpha_1 (x_1 - x_0) + \alpha_2(x_2 - x_0).$$

Optimal portfolio solves the following program (P):

$$\max_{\alpha_1,\alpha_2} \int_{\underline{x}_1}^{\overline{x}_1} \int_{\underline{x}_2}^{\overline{x}_2} u\left(W(\alpha_1, \alpha_2)\right) dF\left(x_1/x_2\right) dG\left(x_2\right)$$

where $[\underline{x}_1, \overline{x}_1]$ and $[\underline{x}_2, \overline{x}_2]$ are respectively the support of $\tilde{x}_1$ and $\tilde{x}_2$ and u is the von Newman Morgenstern utility function, with $u'(.) > 0$ and $u''(.) < 0$.

Assume we have interior solutions, the first order conditions associated to the above program are:

$$\int_{\underline{x}_1}^{\overline{x}_1} \int_{\underline{x}_2}^{\overline{x}_2} (x_1 - x_0)\, u^{'}\left(W(\alpha_1, \alpha_2)\right) dF\left(x_1/x_2\right) dG\left(x_2\right) = 0, \tag{1}$$

$$\int_{\underline{x}_1}^{\overline{x}_1}\int_{\underline{x}_2}^{\overline{x}_2}(x_2-x_0)\,u'\left(W(\alpha_1,\alpha_2)\right)dF\left(x_1/x_2\right)dG\left(x_2\right)=0. \tag{2}$$

In the case of independence, condition (1) evaluated at $\alpha_1 = 0$ can be written as

$$\left(E\left(\widetilde{x}_1\right)-x_0\right)\int_{\underline{x}_2}^{\overline{x}_2}u'\left(\alpha_2\left(x_2-x_0\right)+1+x_0\right)dG\left(x_2\right)$$

which has the sign of the expected excess return $\left(E\left(\widetilde{x}_1\right)-x_0\right)$. It follows that α_1 is positive if and only if $E\left(\widetilde{x}_1\right)-x_0$ is also positive, that is if and only if $\widetilde{x}_1$ offers a positive risk premium. The same comment applies for α_2.

3 Shifts in Returns Distribution of One Risky Asset

We design shifts in one distribution by a partial derivative with respect to the parameter r. So we define $dF(x_1/x_2,r)dG(x_2)$ as the risky initial situation.

We first have the next result for a FSD deterioration in the returns distribution of x_1.

Proposition 1 *Assume that the utility function exhibits constant relative risk aversion and let α_1^* and α_2^* represent optimal investment in the risky assets. Then either $\frac{\alpha_1^*}{\alpha_2^*}$ or α_1^* is decreasing following a FSD deterioration in the returns distribution of $\widetilde{x}_1$.*

Proof

Differentiating the first order condition (2) with respect to r yields:

$$\begin{aligned}&\int_{\underline{x}_1}^{\overline{x}_1}\int_{\underline{x}_2}^{\overline{x}_2}\left(x_2-x_0\right)^2u''\left(W(\alpha_1^*,\alpha_2^*)\right)dF\left(x_1/r\right)dG\left(x_2\right)\frac{d\alpha_2^*}{dr}\\&+\int_{\underline{x}_1}^{\overline{x}_1}\int_{\underline{x}_2}^{\overline{x}_2}\left(x_2-x_0\right)\left(x_1-x_0\right)u''\left(W(\alpha_1^*,\alpha_2^*)\right)dF\left(x_1/r\right)dG\left(x_2\right)\frac{d\alpha_1^*}{dr}\\&+\int_{\underline{x}_1}^{\overline{x}_1}\int_{\underline{x}_2}^{\overline{x}_2}\left(x_2-x_0\right)u'\left(W(\alpha_1^*,\alpha_2^*)\right)dF_r\left(x_1/r\right)dG\left(x_2\right)=0.\end{aligned} \tag{3}$$

The second term in equation (3) can be rewritten as:

$$\int_{\underline{x}_1}^{\overline{x}_1}\int_{\underline{x}_2}^{\overline{x}_2}\left(x_2-x_0\right)\left(x_1-x_0\right)\frac{u''\left(W(\alpha_1^*,\alpha_2^*)\right)}{u'\left(W(\alpha_1^*,\alpha_2^*)\right)}u'\left(W(\alpha_1,\alpha_2)\right)dF\left(x_1/r\right)dG\left(x_2\right). \tag{4}$$

By the assumption of constant relative risk aversion (CRRA) we have:

$$\begin{aligned}(x_1 - x_0)\frac{u''(W(\alpha_1^*, \alpha_2^*))}{u'(W(\alpha_1^*, \alpha_2^*))} &= \frac{\gamma}{\alpha_1^*} - \frac{\alpha_2^*}{\alpha_1^*}(x_2 - x_0)\frac{u''(W(\alpha_1^*, \alpha_2^*))}{u'(W(\alpha_1^*, \alpha_2^*))} \\ &\quad - \frac{1+x_0}{\alpha_1^*}\frac{u''(W(\alpha_1^*, \alpha_2^*))}{u'(W(\alpha_1^*, \alpha_2^*))}, \qquad (5)\end{aligned}$$

where $\gamma > 0$ is the measure of constant relative risk aversion $-W(\alpha_1^*, \alpha_2^*)\frac{u''(W(\alpha_1^*, \alpha_2^*))}{u'(W(\alpha_1^*, \alpha_2^*))}$.

Substituting (5) in (4), we get, after simplification:

$$\begin{aligned}&\frac{\gamma}{\alpha_1^*}\int_{\underline{x}_2}^{\overline{x}_2}\int_{\underline{x}_1}^{\overline{x}_1}(x_2 - x_0)\,u'(W(\alpha_1^*, \alpha_2^*))\,dF(x_1/r)\,dG(x_2) \\ &-\frac{\alpha_2^*}{\alpha_1^*}\int_{\underline{x}_2}^{\overline{x}_2}\int_{\underline{x}_1}^{\overline{x}_1}(x_2 - x_0)^2\,u''(W(\alpha_1^*, \alpha_2^*))\,dF(x_1/r)\,dG(x_2) \\ &-\frac{1+x_0}{\alpha_1^*}\int_{\underline{x}_2}^{\overline{x}_2}\int_{\underline{x}_1}^{\overline{x}_1}(x_2 - x_0)\,u''(W(\alpha_1^*, \alpha_2^*))\,dF(x_1/r)\,dG(x_2)\,. \qquad (6)\end{aligned}$$

The first term in (6) is nil by the first order condition associated to the choice of α_2. (3) can now be rewritten as:

$$\begin{aligned}&\int_{\underline{x}_1}^{\overline{x}_1}\int_{\underline{x}_2}^{\overline{x}_2}(x_2 - x_0)^2\,u''(W(\alpha_1^*, \alpha_2^*))\,dF(x_1/r)\,dG(x_2)\left[\frac{d\alpha_2^*}{dr} - \frac{\alpha_2^*}{\alpha_1^*}\frac{d\alpha_1^*}{dr}\right] \\ &-\frac{1+x_0}{\alpha_1^*}\int_{\underline{x}_2}^{\overline{x}_2}\int_{\underline{x}_1}^{\overline{x}_1}(x_2 - x_0)\,u''(W(\alpha_1^*, \alpha_2^*))\,dF(x_1/r)\,dG(x_2)\,\frac{d\alpha_1^*}{dr} \\ &+\int_{\underline{x}_1}^{\overline{x}_1}\int_{\underline{x}_2}^{\overline{x}_2}(x_2 - x_0)\,u'(W(\alpha_1^*, \alpha_2^*))\,dF_r(x_1/r)\,dG(x_2) = 0.\end{aligned}$$

We now prove that

$$\int_{\underline{x}_1}^{\overline{x}_1}\int_{\underline{x}_2}^{\overline{x}_2}(x_2 - x_0)\,u'(W(\alpha_1^*, \alpha_2^*))\,dF_r(x_1/r)\,dG(x_2) \leq 0 \qquad (7)$$

under a FSD deterioration in the returns of $\widetilde{x}_1$.

We know from Eeckhoudt, Gollier and Schlesinger [1996] that a FSD deterioration in an independent background risk would make an individual with a decreasing absolute risk aversion behave in a more risk-averse way and hence decrease his exposure to the other independent risk (in our model he reduces α_2). Here, decreasing absolute risk aversion assumption is verified since we are in the class of CRRA utility functions. To complete the proof we interpret $\alpha_1^*(\tilde{x}_1 - x_0)$ as a background risk. Let's note its distribution function as η. We have that

$$d\eta\left(\varepsilon/r\right) = \alpha_1 dF\left(\frac{\varepsilon}{\alpha_1^*} + x_0/r\right).$$

As we can see, if

$$F\left(\varepsilon/r + h\right) \geq F\left(\varepsilon/r\right) \text{ for all } \varepsilon \text{ and all } h > 0,$$

then

$$\eta\left(\varepsilon/r + h\right) \geq \eta\left(\varepsilon/r\right) \text{ for all } \varepsilon \text{ and all } h > 0.$$

It follows from the discussion made earlier that

$$\int_{\underline{x}_1}^{\overline{x}_1} \int_{\underline{\varepsilon}}^{\overline{\varepsilon}} \left(x_2 - x_0\right) u^{'}\left(1 + x_0 + \alpha_2^*(x_2 - x_0) + \varepsilon\right) d\eta\left(\varepsilon/r\right) dG\left(x_2\right) = 0$$

and

$$\int_{\underline{x}_1}^{\overline{x}_1} \int_{\underline{\varepsilon}}^{\overline{\varepsilon}} \left(x_2 - x_0\right) u^{'}\left(1 + x_0 + \alpha_2^*(x_2 - x_0) + \varepsilon\right) d\eta\left(\varepsilon/r + h\right) dG\left(x_2\right) \leq 0$$

so that for all $h > 0$,

$$\begin{aligned} 0 &= \int_{\underline{x}_1}^{\overline{x}_1} \int_{\underline{x}_2}^{\overline{x}_2} \left(x_2 - x_0\right) u^{'}\left(W(\alpha_1^*, \alpha_2^*)\right) dF\left(x_1/r\right) dG\left(x_2\right) \\ &\geq \int_{\underline{x}_1}^{\overline{x}_1} \int_{\underline{x}_2}^{\overline{x}_2} \left(x_2 - x_0\right) u^{'}\left(W(\alpha_1^*, \alpha_2^*)\right) dF\left(x_1/r + h\right) dG\left(x_2\right), \end{aligned}$$

or

$$\int_{\underline{x}_1}^{\overline{x}_1} \int_{\underline{x}_2}^{\overline{x}_2} \left(x_2 - x_0\right) u^{'}\left(W(\alpha_1^*, \alpha_2^*)\right) \left[\frac{dF\left(x_1/r + h\right) - dF\left(x_1/r\right)}{h}\right] dG\left(x_2\right) \leq 0.$$

Allowing h to approach 0 we obtain the inequality in (7).

It remains to show that

$$\int_{\underline{x}_1}^{\overline{x}_1} \int_{\underline{x}_2}^{\overline{x}_2} \left(x_2 - x_0\right) u^{''}\left(W(\alpha_1^*, \alpha_2^*)\right) dF\left(x_1/r\right) dG\left(x_2\right) \geq 0. \tag{8}$$

In fact, since the utility function is CRRA then it is necessary a mixed risk aversion utility function (for more details and properties of mixed risk aversion utility functions see Caballé and Pomansky, 1996; Dachraoui et al., 1999; Brockett and Golden, 1996) which means that its first derivative is a complete monotone function. Let *dU(.)* be the measure describing the mixture of exponential utilities. We can now write (8) as

$$-\int_0^\infty t\left[\int_{\underline{x}_2}^{\overline{x}_2} e^{-t\left(1+x_0+\alpha_1^*(x_1-x_0)\right)}dF\left(x_1/r\right)\right]\left[\int_{\underline{x}_1}^{\overline{x}_1}\left(x_2-x_0\right)e^{-t\alpha_2^*(x_2-x_0)}dG\left(x_2\right)\right]dU\left(t\right).$$

From the first order condition (2) we can show that there exists t^* such that

$$\int_{\underline{x}_1}^{\overline{x}_1}\left(x_2-x_0\right)e^{-t\alpha_2^*(x_2-x_0)}dG\left(x_2\right)\geq 0 \text{ if and only if } t\leq t^*.$$

Moreover, since

$$\int_{\underline{x}_2}^{\overline{x}_2} e^{-t\left(1+x_0+\alpha_1^*(x_1-x_0)\right)}dF\left(x_1/r\right)\geq 0 \text{ for all } t,$$

it follows that

$$\begin{aligned}
& t\int_{\underline{x}_1}^{\overline{x}_1}\left(x_2-x_0\right)e^{-t\alpha_2^*(x_2-x_0)}dG\left(x_2\right)\int_{\underline{x}_2}^{\overline{x}_2} e^{-t\left(1+x_0+\alpha_1^*(x_1-x_0)\right)}dF\left(x_1/r\right) \\
\leq\ & t^*\int_{\underline{x}_1}^{\overline{x}_1}\left(x_2-x_0\right)e^{-t\alpha_2^*(x_2-x_0)}dG\left(x_2\right)\int_{\underline{x}_2}^{\overline{x}_2} e^{-t\left(1+x_0+\alpha_1^*(x_1-x_0)\right)}dF\left(x_1/r\right).
\end{aligned}$$

Integrating over t gives

$$\begin{aligned}
& \int_{\underline{x}_1}^{\overline{x}_1}\int_{\underline{x}_2}^{\overline{x}_2}\left(x_2-x_0\right)u''\left(W\left(\alpha_1^*,\alpha_2^*\right)\right)dF\left(x_1/r\right)dG\left(x_2\right) \\
\geq\ & -t^*\int_{\underline{x}_1}^{\overline{x}_1}\int_{\underline{x}_2}^{\overline{x}_2}\left(x_2-x_0\right)u''\left(W\left(\alpha_1^*,\alpha_2^*\right)\right)dF\left(x_1/r\right)dG\left(x_2\right) \\
=\ & 0.
\end{aligned}$$

Consequently in order to have the equality in equation (3) we necessarily have that either $\frac{\alpha_1^*}{\alpha_2^*}$ or α_1^* is decreasing in r which ends the proof of Proposition 1. *Q.E.D.*

We now consider a SSD deterioration in return on the first asset. We say that a utility function u exhibits a decreasing absolute risk aversion in the sense of Ross over the relative range of wealth if there exists a scalar λ such that

$$p\,(w+y) \geq \lambda \geq r\,(w+y')\,,\ \forall\ y,\ y' \in [a,b]\,,$$

where $p\,(\ .)$ stands for absolute prudence and $r\,(.)$ for absolute risk aversion.

We can now show the next result:

Proposition 2 *Let the utility function u exhibit constant relative risk aversion and suppose that it has an absolute risk aversion that is decreasing in the sense of Ross, then the result in Proposition 1 holds following a SSD deterioration on the returns distribution of $\widetilde{x}_1$.*

Proof.

The proof is similar to that of Proposition 1 except that we need a stronger condition for the sign of

$$\int_{\underline{x}_1}^{\overline{x}_1}\int_{\underline{x}_2}^{\overline{x}_2}(x_2-x_0)\,u^{'}\,(W(\alpha_1^*,\alpha_2^*))\,dF_r\,(x_1/r)\,dG\,(x_2)\,.$$

For a SSD we observe that

$$\int_{\underline{\varepsilon}}^{\overline{\varepsilon}}\eta\,(\varepsilon/r+h) \geq \int_{\underline{\varepsilon}}^{\overline{\varepsilon}}\eta\,(\varepsilon/r)\ \text{ for all } h>0,$$

whenever

$$\int_{\underline{\varepsilon}}^{\overline{\varepsilon}}F\,(\varepsilon/r+h) \geq \int_{\underline{\varepsilon}}^{\overline{\varepsilon}}F\,(\varepsilon/r)\ \text{ for all } h>0.$$

We conclude that

$$\int_{\underline{x}_1}^{\overline{x}_1}\int_{\underline{x}_2}^{\overline{x}_2}(x_2-x_0)\,u^{'}\,(W(\alpha_1^*,\alpha_2^*))\,dF_r\,(x_1/r)\,dG\,(x_2) \leq 0$$

following Corollary 1 in Eeckhoudt, Gollier and Schlesinger [1996]. *Q.E.D.*

4 Concluding remarks

Propositions 1 and 2 show that a FSD or a SSD deterioration that affects the distribution of x_1 will reduce the weight of this asset in the optimal fund. This contraction may reduce both α_1^* and α_2^* but the relative effect on α_1^* is more important. It should be notified that $\frac{\alpha_2^*}{\alpha_1^*}$ is increasing in r for all $u(\cdot)$ that are $CRRA$ and whatever the level of risk aversion. This means that the two-fund separation theorem holds for all r since $CRRA$ functions are in the class of utility functions that permit mutual-fund separation. The additional restrictions for a SSD deterioration is to yield a particular direction on the variation of the ratio $\frac{\alpha_2^*}{\alpha_1^*}$. Consequently, when the two-fund conditions hold, following a shift in the returns distribution, the investor must first evaluate the variation in the proportions of the assets in the risky fund and then decide how to redivide his total wealth between the two funds.

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