Optimal Cognitive Processes for Lotteries

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Abstract

The aim of the present paper is to propose a rational model of decision-making for lotteries. The key element of the theory is the use of cognitive processes. The maximization of the degree of confidence associated with each judgment involves different processes. Our contribution explains some major violations of the expected-utility theory for decisions on lotteries.

Key words: Cognitive process, lottery, paradox, bounded rationality, preference reversal, common ratio.
JEL Classification: D80.

Résumé

Le but de cette recherche est de proposer un modèle rationnel de prise de décision sur les loteries. L'élément clé de la théorie est l'utilisation d'un processus cognitif. La maximisation du degré de confiance associé à chaque jugement implique différents processus. Notre contribution explique quelques défaillances importantes de la théorie de l'espérance d'utilité sur les choix de loterie.

Mots clés : Processus cognitif, loterie, paradoxe, rationalité limitée, renversement des préférences, ratio commun.
Classification JEL : D80.

INTRODUCTION

Two important facts related to decision-making under risk are the following: First, there exist many cognitive anomalies that may be consistent with rationality at some level of abstraction, such as asymmetry, certainty, ... (see McFadden, 1999, for a complete list). Preference reversal and other paradoxes, (Machina, 1987, Camerer, 1992) belong to this family of cognitive anomalies. Second, we may observe from different tests many restructuring operations (grouping by similarity, cancellation, framing ...) (Ranyard, 1995).

The aim of this article is to propose a decision model that explains how and why rational individuals who behave optimally use these different restructuring operations to make decisions that are coherent with test results for lotteries. Many results of these tests can be interpreted as cognitive anomalies.

The intuition behind the model is that the judgment between two elements is never perfect and depends on the likeness between the elements. Since the individual have limited computational ability, as in Payne et al. (1993) for example, he tries to maximize his understanding of the problem which leads him to use the most appropriate cognitive process.

The basic concept of the proposed theory is that lottery elements have qualities. This idea is not new. For example Kahneman and Tversky (1979) frame outcomes as gains and losses. They consider that a loss looms larger than the corresponding gain and that there exists a difference between negative and positive qualities for the monetary amounts. Prelec (1998) mentions the qualitative character given by the transition from impossibility ($p_i = 0$) to possibility or risk ($p_i \in]0,1[$), while Kahneman and Tversky (1979) make a similar remark when considering the difference between certainty ($p_i = 1$) and risk. In this paper, we will consider that the probabilities $p_i = 0$ or 1 are qualitatively different from the probabilities $p_i \in]0,1[$. Some of the qualities discussed are already mentioned in the above contributions. Others are defined and introduced in Section 1.

We will consider the concept of quality as in the above-mentioned articles but also in another way not found in the literature. We can condense all facts about processes into one simple principle: the existence of a degree of confidence in the judgment. When the elements are very different, the judgment is difficult to make and then the degree of confidence is weak. For example, if an individual compares a positive monetary amount x_1 to a negative one $-x_2$, the degree of confidence of this judgment is weaker than the one that considers two positive monetary amounts x_1 and x_2. Hence, an agent who wants to maximize the degree of confidence will compare elements that are the most similar. This maximization involves the use of different processes. Other processes proposed in this paper are presented in examples 2.1, 2.2, and 2.3 below.

Consequently, the concept of qualities helps the process in two different ways. First, the maximization of the degree of confidence settles the process and, second, the qualities are taken explicitly into account in the judgment. This procedure will involve changes in the qualities considered for the probabilities and will be able to explain the existence of common ratio and preference reversal paradoxes.

The main goal of this article is to explore an approach where rational processes are taken into account explicitly. In Section 1, we propose a definition of lottery qualities and build up a simple model which is, however, sufficient to illustrate how the principles work. Our major contribution is in Section 2 where we propose a way to obtain maximal rational processes. In Section 3, we apply the model to explain five basic tests that correspond to the two above-mentioned paradoxes: Three lottery choices and two pricing decisions for lotteries where the prices are "judged-certainty-equivalents" (JCE).

1. BASIC CONCEPTS

1.1 Definitions

We employ the notation {a,b} to define a set and the notation (a,b) to define an ordered pair. A monetary amount that has values in $[0,\infty[$ is denoted x_i and $x_i \in X$. A probability that has values in [0,1] is denoted p_i and $p_i \in P$. We limit the analysis to two-point lotteries having a positive monetary amount x_i with probability $p_i \in]0,1[$ and 0 with probability $1-p_i$. So a non-degenerate lottery is written (p_i,x_i).

The sets P and X represent different qualities of lotteries. These sets could be split into more refined sets associated with more refined qualities.

As in the literature (Kahneman and Tversky, 1979, Prelec, 1998), we assume that the set P is split into a set S composed of elements with the "surety" quality and a set R comprising elements with the "risk" quality, so we define a refinement (or partition) of P as $\Re_P = \{S,R\}$. The elements $p_i \in \{0,1\}$ have the "surety" quality and the elements $p_i \in]0,1[$ have the "risk" quality. We also add two other qualities by splitting the set R into two parts so we can define $\Re_R = \{W,L\}$, where the elements of W have the "winning" quality and the elements of L have the "losing" quality. So the elements $p_i \in]0,p'[$ have the "losing" quality and the elements $p_i \in [p',1[$ have the "winning" quality. Kagel et al. (1990), in a test on the common-ratio paradox, obtain that, for lotteries with probabilities lower than .2, the lottery with the smaller probability is chosen. So the probability frontier between sets L and W should be greater than .2. If we now consider the pricing decision as in Tversky and Kahneman (1992), this frontier point has to be the fixed point with a value of about .35. This fixed point can represent our p', but any value higher than .2 and lower than .5 is acceptable according to different tests. The parameter β in the function $w(p) = e^{-\beta(-\ln p)^{\alpha}}$ (Prelec, 1998) permits the existence of different p'. Figure I illustrates the above refinements of sets P and R:

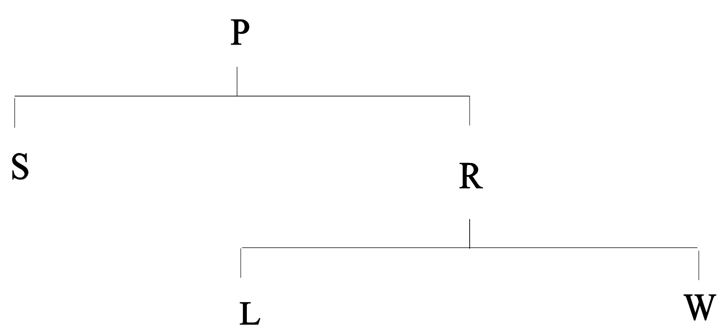

Figure I: Refinements of sets P and R

For example, the probability $.8 \in W \subset R \subset P$; moreover $1 \in S \subset P$. We do not define any refinement for X since we want to concentrate our analysis of the individual's choice on his probability judgments.[1] Thus, for our purpose, we have a collection of sets $\Gamma = \{P,X,S,R,W,L\}$ and we denote, for example, $P \in \Gamma$ to identify P as an element of Γ. We will use Q_j and Q_ℓ to denote elements of Γ and $Q_\ell \times Q_j$ for an ordered pair.

1.2 Model

1.2.1 Lottery choice

By using the definition of p' we assume that there exist three possible tests for lottery choices (p_1,x_1) vs (p_2,x_2) where $p_1,p_2 \in]0,1[$ and $x_1,x_2 > 0$. The first one is where the two probabilities are high: $p_1,p_2 \geq p'$. The second one is where the two probabilities are low: $p_1,p_2 < p'$. A third test is for a lottery where one probability is high ($p_1 \geq p'$) and the other one is low ($p_2 < p'$). An evaluation function that takes these three cases into account is represented by the next equation where $p_1\text{-}p_2 = \Delta p$, $p_1+p_2 = \Sigma p$, $x_1\text{-}x_2 = \Delta x$, $x_1+x_2 = \Sigma x$ and $\alpha_{Q_\ell \times Q_j}$ is a weighting parameter that measures the perceived difference between the probabilities p_1 and p_2:

$$C\left(p_1,x_1,p_2,x_2\right) = \alpha_{Q_\ell \times Q_j}\ \Delta p\ \Sigma x/2 + \Delta x\ \Sigma p/2. \tag{1}$$

Consequently, $\alpha_{Q_\ell \times Q_j} = \alpha_{W \times W}$ if $p_1, p_2 \geq p'$, $\alpha_{Q_\ell \times Q_j} = \alpha_{L \times L}$ if $p_1, p_2 < p'$ and

$\alpha_{Q_\ell \times Q_j} = \alpha_{W \times L}$ if $p_1 \geq p' > p_2$. $C\left(p_1, x_1, p_2, x_2\right)$ can be interpreted as a measure of

the perceived difference between the two lotteries.

Observe that when $\alpha_{Q_\ell \times Q_j} = 1$, equation (1) is equivalent to:

$$\left(p_1 - p_2\right)\left(x_1 + x_2\right)/2 + \left(x_1 - x_2\right)\left(p_1 + p_2\right)/2 = p_1 x_1 - p_2 x_2 .$$

For $p \in\ ]0,1[$, Rule 6 in MacCrimmon and Larsson (1979) states: "when one alternative provides an almost sure chance of obtaining a very desirable consequence, select it, even if it entails passing up a larger amount having a lower probability. When, however, the chances of winning are small and close together, take the option that provides the larger payoff." Consequently, this rule suggests that $\alpha_{W \times W} > 1$ for high probabilities: $p_i \geq p'$; and $\alpha_{L \times L} < 1$ for small probabilities $p_i < p'$. It is natural to think that the “winning” quality will be preferred to the “losing” quality for most people, so $\alpha_{W\times L} > 1$. Since for the judgment of monetary amounts there is no difference between qualities, this is equivalent to the idea of Slovic and Lichtenstein (1983) who consider that choices among pairs of gambles appear to be influenced primarily by probabilities of winning and losing rather than by dollar amounts in the paradox-preference reversal.

We observe that equation (1) implies the judgment of both probabilities together $\left(\alpha_{Q_\ell \times Q_j}\ \Delta p\right)$ and both monetary amounts together (Δx). This seems to be a natural way of judging elements p_i and x_i. Rubinstein (1988) and Leland (1994) examine this procedure. Moreover, some tests with verbal reports seem to confirm this procedure. An example of a verbal report from a subject is the following (Ranyard 1995): "There's more chance of winning a smaller amount..." It indicates clearly that the subject compares the two probabilities together. The model we propose in this paper shows that this rational procedure is optimal for lottery choices. Let us consider two numerical examples of equation (1).

Example 1.1

A test that corresponds to the comparison of two lotteries in the preference reversal paradox found in Tversky et al. (1990) is (.97,4) vs (.31,16) where 83% of the subjects choose (.97,4). This result can be obtained by using equation (1):

$\Delta p = .66$, $\Delta x = -12$, $\Sigma x/2 = 10$ and $\Sigma p/2 = .64$;

So $C(.97,4,.31,16) = \alpha_{W\times L}\ (.66)\ 10 + (-12)\ .64$;

$C(.97,4,.31,16) = \alpha_{W\times L}\ 6.6 - 7.68$.

When $\alpha_{W \times L} \geq 1.164$, that is when the quality of W is preferred to the quality of L, the first lottery is chosen. Note that when the qualities play no role ($\alpha_{W \times L} = 1$), C(.97,4,.31,16) = -1.08 and represents the difference $p_1 x_1 - p_2 x_2$. So the second lottery is chosen.

<u>Example 1.2</u>

Problems 7 and 8 in Kahneman and Tversky (1979) are examples of common ratio. In Problem 7, 86% of the subjects choose the lottery (.90,3000) when it is compared to (.45,6000). For this case equation (1) can be written as:

$$\begin{aligned} C(.9,3000,.45,6000) &= \alpha_{W \times W}\, .45\ (4500) - .675\ (3000) \\ &= \alpha_{W \times W}\, 2025 - 2025 \end{aligned}$$

and $\alpha_{W \times W} > 1$ explains the result. In Problem 8, 73% of the subjects choose the lottery (.001,6000) when it is compared to (.002,3000):

$$C(.002,3000,.001,6000) = \alpha_{L \times L}\, 4.5 - 4.5$$

and $\alpha_{L \times L} < 1$ explains this result. Both correspond to Rule 6 in MacCrimon and Larsson (1979).

1.2.2 Pricing

The preference-reversal paradox shows that the result of the comparison of two lotteries differs from that of their pricing using "judged-certainty-equivalent" (JCE). So when the individual must reveal his JCE, the model must be different to that in section 1.2.1. Note that the JCE is different from the "choice-certainty-equivalent". See Luce (2000) for a recent detailed comparison of the two concepts.

In this paper we assume that the individual will judge the p and x values with the boundary associated with each set. The boundaries B_k for p are $B_1 = 0$ and $B_2 = 1$ and the boundary for x is 0. So the JCE value for pricing is:

$$P\left(p_1, x_1\right) = \left[B_k + \alpha_{S\times R}\left(p_1 - B_k\right)\right] \times \left[0 + \left(x_1 - 0\right)\right] \qquad (2)$$

where $B_k = B_2$ when $p_i \geq p'$, and $B_k = B_1$ when $p_i < p'$ and p' is the same as for equation (1)[2].

Note the symmetry between the expressions in the square brackets of (2). It indicates that p_i and x_i are judged in the same manner, in the sense that the value is first compared with the boundary and then added to this boundary. The parameter $\alpha_{S\times R} > 1$ weights the qualitative difference between S (surety) and R (risk). If qualities play no role ($\alpha_{S\times R} = 1$), equation (2) is equal to $p_i x_i$. Notice that the probabilities p_i are evaluated either with the boundary 0 or the boundary 1, depending on whether p_i is below or above the probability frontier p'. It is natural that a rational individual would use the same probability frontier for the comparison and pricing of lotteries. Let us now consider a pricing example.

<u>Example 1.3</u>

Birnbaum et al. (1992) obtain a JCE of about 70 dollars for (.95,96) . From equation 2 we have

$$\begin{aligned} P(.95,96) &= (1-\alpha_{S\times R}\,.05)\ 96 \\ &= 96 - 4.8\ \alpha_{S\times R}. \end{aligned}$$

For $\alpha_{S\times R} \approx 5.4$ we obtain $P(.95,96) \approx 70$. This result shows that the perceived difference between the elements of S and R proportional to $\alpha_{S\times R}$ is much greater than the perceived difference between the elements W and L proportional to $\alpha_{W\times L}$ as in Example 1.1.

Some finer details must be discussed concerning the models. For the JCE, (equation 2), we can also consider a parameter $\alpha_{Q_\ell \times Q_j} \neq \alpha_{S\times R}$ that is proportional to the distance from the boundary and measured by [1+(p'-p)] for p < p' and [1-(p'-p)] for p ≥ p'. This case yields a regressive inverse S-shaped w(p) function as in Prelec (1998)[3]. So $P(p_1,x_1) = w(p_i)x_i$. It should be noted that for the pricing of lotteries, the probability weighting function can be represented by a function with one argument even if it is derived from the judgment of the probability with a boundary. However the need for a function with two arguments appears clearly in equation (1) where the judgment of $p_1 \in W$ differs when $p_2 \in W$ ($\alpha_{Q_\ell \times Q_j} = \alpha_{W\times W}$) or $p_2 \in L$ ($\alpha_{Q_\ell \times Q_j} = \alpha_{W\times L}$). (See Alarie and Dionne, 2001, for details.)

For equation (1) the value of the parameter $\alpha_{W\times L}$ could be proportional to the difference between the probabilities. This approach is equivalent to splitting the set of probabilities into an infinite number of qualities. Another plausible approach is to divide the set of probabilities into three sets: low, medium, and high. These specifications do not affect the usefulness for equations (1) and (2) for our purpose.

1.2.3 Three Questions

Although the contributions of MacCrimon and Larsson (1979), Slovic and Lichtenstein (1983), Tversky and Kahneman (1979), Prelec (1998) and Leland (1994) seem to explain well the individual's choice of lotteries (p_i,x_i), three questions remain unanswered. First, why does the individual judge the two probabilities together and the two monetary amounts together? We will show that this is an optimal process.

A second question is: Why and how does the individual choose the qualities to be considered? For example, when he has to decide between two lotteries $(.8, x_1)$ and $(.9, x_2)$

where $.8 \in W \subset R \subset P$ and $.9 \in W \subset R \subset P$ then he must choose which pair of qualities ($W{\times}W, W{\times}R, W{\times}P, R{\times}W, \ldots$) matters. We will also show that this decision is a step in an optimal process.

The third question is related to the preference-reversal paradox in which subjects provide both “judged-certainty-equivalents” for lotteries and choices between pairs of them. For many subjects, lottery pairs exist for which the order established by the “judged-certainty-equivalents” is opposite to that of the lottery choice (Bostic et al., 1990). Consequently, the paradox implies that the process used to obtain the “judged-certainty-equivalent” is different from the one used in the lottery choice. Note also that, for the type of lotteries used in this paper, the paradox exists for the “choice-certainty-equivalent” (Alarie and Dionne, 2001).

Moreover, in the lottery choice, the subject could obtain the “judged-certainty-equivalent” of each lottery and then compare them. But the test results (Bostic et al., 1990, Tversky et al., 1990) imply that he does not use this process. In conclusion, the process for lottery choices is preferred to the one that yields the “judged-certainty-equivalents”. We must then explain why subjects have preferences as regards processes. This is equivalent to answering the following question: Why doesn’t the agent price the lotteries when making his lottery choice? We will show that is not an optimal process.

2. A RATIONAL THEORY

2.1 Judgment function

In the preceding section we introduced two equations representing different judgments of lottery elements based on their respective qualities. For example, for the judgments of two probabilities, the parameter $\alpha_{Q_\ell \times Q_j}$ in equation (1) has different values for probabilities belonging to different sets (W,L,S,R). In the same manner, the judgment of x_i uses the set X×X, since x_i belongs to only one possible set.

A general definition of these judgments where θ_j and θ_ℓ can represent either a monetary amount, a probability or the result of a judgment is the following:

DEFINITION 1: *Let $Q_\ell \times Q_j \in \Gamma \times \Gamma$. Let θ_ℓ, θ_j be two elements to be judged. For each pair $Q_\ell \times Q_j$ corresponds a judgment function $J_{Q_\ell \times Q_j}(\theta_\ell, \theta_j)$: $R^1 \times R^1 \rightarrow R^1$.*

With the notation of Definition 1, equation (1) becomes:

$$C\left(p_1, x_1, p_2, x_2\right) = J_{P \times X}\left(J_{Q_\ell \times Q_j}\left(p_1, p_2\right), J_{X \times X}\left(x_1, x_2\right)\right) \qquad (1')$$

For example for $p_1, p_2 < p'$ where the decision-maker uses $\alpha_{L \times L}$ in equation 1, $J_{Q_\ell \times Q_j}(p_1,p_2) = J_{L \times L}(p_1,p_2)$. So equation (1) is simply the sum of the value of the judgment of the two p_i weighted by $\Sigma p/2$ and the value of the judgment of the two x_i weighted by $\Sigma x/2$. Consequently, equation (1) represents a process having three basic judgments. From now on, the discussion will relate to the general form (1') which emphasizes that the model to be developed focuses on the qualities associated with each judgment and not on a particular function such as equation (1). However, this particular form can be useful for empirical tests. A similar exercise can be made for equation (2). So we obtain:

$$P\left(p_1, x_1\right) = J_{P \times X}\left(J_{S \times R}\left(B_k, p_i\right), J_{X \times X}\left(x_1, 0\right)\right) \qquad (2')$$

where the judgment of a probability with the boundaries is represented by $J_{S\times R}(0,p_1)$ when $p < p'$ or by $J_{S\times R}(1,p_i)$ otherwise. Finally if $J_{Q_\ell \times Q_j}(\theta_\ell,\theta_j) = \theta_n$, then θ_n belongs to the set $Q_n = Q_\ell \cup Q_j$ and to less refined qualities Q_m such that $Q_n \subset Q_m$. For example, the judgment of two x_i belongs to $X \cup X = X$. Another example with less refined qualities (R, P) concerning the judgment of two probabilities is $J_{W\times L}(p_1,p_2)$ where the result belongs to $W \cup L = R \subset P$.

2.2 Axioms

The three following axioms will be useful for the elimination of judgments that cannot correspond to the different tests in the literature.

(A1) *If $J_{Q_\ell \times Q_j}(\theta_\ell, \theta_j)$ exists then $\theta_\ell \in Q_\ell$ and $\theta_j \in Q_j$.*

This first axiom means that if the agent is able to make a judgment then the element θ_ℓ belongs to the set Q_ℓ and θ_j belongs to the set Q_j . For example, this axiom eliminates the case where an individual tries to judge the probabilities $.1 < p'$ and $.8 \geq p'$ together with the quality of $W\times W$. This makes no sense because .1 does not belong to W. So the individual cannot feel he has a winning lottery while he is judging .1. The second axiom is for situations where individuals consider refinements of sets[4].

(A2) *Let $\mathfrak{R}_n$ be an element of $\{\mathfrak{R}_P, \mathfrak{R}_R,\}$ and $Q_\ell \in \mathfrak{R}_n$ or $Q_j \in \mathfrak{R}_n$. If $J_{Q_\ell \times Q_j}(\theta_\ell, \theta_j)$ exists then Q_ℓ, $Q_j \in \mathfrak{R}_n$.*

The judgment function considers only one refinement and then limits the number of possible comparisons. For example suppose a situation where the probability $p_1=1 \in S$ and the probability $p_2=.8 \in W \subset R$. If the individual tries to judge them by considering the qualities of S and W, this type of judgment is not possible in our model because the qualities come from two refinements $\Re_P = \{R,S\}$ and $\Re_R = \{W,L\}$. In this example, the individual is able to consider the difference between the quality of R and the quality of S but he is not able to consider the quality of W and that of S together.

The last axiom is about judgments with boundaries. If we have already judged two p_i, for example p_1=.6 and p_2=.8, then the perceived difference .2 $\alpha_{Q_\ell \times Q_j}$ cannot be judged with the boundaries 1 or 0. This axiom prevents an infinite number of judgments with the boundaries.

(A3) *Let* $\theta_n = J_{Q_\ell \times Q_j}(\theta_\ell, \theta_j)$*, then* $J_{Q_\ell \times Q_j}(\theta_n, B_k)$ *does not exist* $\forall B_n$.

2.3 Degree of confidence

When an individual compares two lotteries (p_1,x_1) and (p_2,x_2) he has to decide which pair of elements will be judged. For example p_1 can be judged in a first instance with x_1, x_2 or p_2. The individual also has to decide on which qualities the four elements will be judged (for example, two p_i can be judged by considering the qualities of R×S or W×L or W×W). A way to explain this ranking of judgments is to consider that an agent has limited computational abilities (bounded rationality). When the elements are very different, the judgment is more difficult to make and the degree of confidence associated with this judgment is weaker. For example, the judgment of two probabilities in W is better (higher degree of confidence) than that of a probability in W and a probability in L. These two judgments are better than the one with the qualities of R and S (see fig. 1). This way of comparing judgments involves, among other things, the fact that better judgments are associated with more refined qualities. The notion of degree of confidence is close to the definition of accuracy (Payne et al., 1993) applied, however, to the whole process rather than to each judgment, as in this paper.

DEFINITION 2: *Let* $Q_\ell \times Q_j \in \Gamma \times \Gamma$*. For each* $Q_\ell \times Q_j$ *corresponds a degree of confidence* $\tau_{Q_\ell \times Q_j} \in [0,1]$.

From the previous analysis of quality pairs, we derive the following relationships where more refined qualities (see fig. 1) are associated with higher degrees of confidence. For judgment of probabilities

(R1) $\tau_{W\times W} = \tau_{L\times L} > \tau_{W\times L} > \tau_{R\times R} = \tau_{S\times S} > \tau_{S\times R} > \tau_{P\times P}$.

Let us now consider the monetary amounts in X. We have only one degree of confidence ($\tau_{X\times X}$) associated with any pair (x_1,x_2), since for the monetary amounts we do not refine set X. For the judgment of one element of P and one element of X, (since the elements differ more than the ones considered by a judgment of two p_i or a judgment of two x_i), we propose the following relationships between the degrees of confidence:

(R2) $\tau_{P\times P}, \tau_{X\times X} > \tau_{P\times X}$.

R2 is close to Payne et al. (1993) who consider that the product of a probability and a monetary amount is more complex than the difference between two monetary amounts, for example. Definition 3 shows how the degrees of confidence are used to obtain a maximal judgment.

DEFINITION 3: *Let* $A = \{J_{Q_1\times Q_2}(\theta_1,\theta_2), J_{Q_3\times Q_4}(\theta_3,\theta_4), \ldots, J_{Q_\ell\times Q_j}(\theta_\ell,\theta_j), \ldots, J_{Q_L\times Q_J}(\theta_L,\theta_J)\}$ *be a set of judgments. A judgment* $J_{Q_n\times Q_m}(\theta_n,\theta_m)$ *is maximal for A if and only if* $\tau_{Q_n\times Q_m} \geq \tau_{Q_\ell\times Q_j}$ *for all* ℓ, *j.*

Axioms A1, A2, A3 and Relations R1 and R2 permit the use of the right quality at the right moment, which is the natural way to think about lotteries, as in Kahneman and Tversky (1979) or Prelec (1998) for S×R, Slovic and Lichtenstein (1983) for W×L, and rule 6 in MacCrimmon and Larsson (1979) for W×W and L×L. These three approaches are known to solve respectively the pricing of lotteries, the lottery choice in preference reversal, and the common-ratio paradox. Let us consider two examples that make use of axioms A1, A2, A3 and relations R1, R2.

<u>Example 2.1</u>

Let us consider the two lotteries of Example 1: (.97,4) and (.31,16). The judgment of .97 (> p') with .31 (< p') is not possible on W×W by A1 since .31 $\notin$ W and then the maximal degree of confidence for .97 and .31 is obtained with the qualities of W×L by A1, A2, and R1.

<u>Example 2.2</u>

For the “judge-certainty-equivalent” of the lottery (.97,4) since both 0, 1 $\in$ S $\subset$ P, the judgment of .97 $\in$ W $\subset$ R $\subset$ P with either the boundary 0 or the boundary 1 is possible on W×S, W×P, R×P, P×S and R×S or P×P by A1. The first four cases are eliminated by A2 and R×S is maximal by Relation R1 where $\tau_{R\times S} > \tau_{P\times P}$. By analogy, $\tau_{R\times S}$ is also maximal for the probability .31 of the lottery (.31,16) in Example 2.1.

To obtain a maximal process, an agent first chooses one maximal judgment available among all possible judgments of one or two elements and repeats this procedure for the new set and so on.[5] For example, if an individual compares two lotteries (p_1,x_1) and (p_2,x_2), he considers four elements. If the first judgment is with the first two elements, lets say x_1 and x_2, the result J(x_1,x_2) becomes itself an element for the next step. The individual must again make a maximal choice among the new set of three elements p_1, p_2 and J(x_1,x_2). The next definition shows how to obtain a maximal process. Note that axiom A3 allows only the basic elements p_i and x_i to be judged with the boundaries.

DEFINITION 4: *Let $\Theta_0 = \{p_1,x_1, p_2,x_2...p_{I/2},x_{I/2}\}$ and $B = \{\beta_1...\beta_M\}$ a set of boundaries. Let $\theta_{n+I} = J_n(\theta_j,\theta_k)$ such that $\theta_j,\theta_k \in \Theta_{n-1} \cup B$. Let $\Theta_n = \Theta_{n-1} \cup \{\theta_{n+I}\} - \{\theta_p$, $p = j,k$ such that $\theta_p \in \Theta_{n-1}\}$. Define also $\wp(\Theta_n)$ as the set of all possible judgments (satisfying A1, A2 and A3) of either two elements of Θ_n or possible judgments of one element of Θ_n with one boundary. A maximal process is a series of maximal judgments $J_1, ..., J_N$ that ends when there remains only one element left in Θ_N, such that:*

J_1 is a maximal element of $\wp(\Theta_0)$

J_2 is a maximal element of $\wp(\Theta_1)$

"

J_n is a maximal element of $\wp(\Theta_{n-1})$,

"

"

J_N is a maximal element of $\wp(\Theta_{N-1})$.

Then a maximal process consists in choosing at each step a maximal judgment among the judgments considering elements that have not already been judged.

Example 2.3

If an individual compares two lotteries (.97,4) and (.31,16), there are four elements (I=4) in the set $\Theta_0 = \{.97, 4, .31, 16\}$.

If the maximal judgment J_1 in the first step of the maximal process is the one with 4 and 16, and if $J_1(4,16) = \theta_5$, we have the new set of three elements $\{.97, .31, \theta_5\} = \Theta_1$. Then in a second step the individual must choose again a maximal judgment among the new set $\wp(\Theta_1)$.

If the maximal judgment is $J_2(.97,.31) = \theta_6$, he must choose a maximal judgment J_3 in the set $\wp(\Theta_2)$. Consequently, there remains only two elements θ_5 and θ_6 to be judged and if the ending maximal judgment is $J_3(\theta_5,\theta_6)$, then the maximal process can be summarized by $J_3(J_1(4,16), J_2(.97,.31))$.

We have seen that a maximal process is a series of maximal judgments where the number of possible judgments are limited by the axioms A1, A2, and A3. The ranking of the $\tau_{Q_\ell \times Q_j}$ is done with the help of the relations R1 and R2.

3. RESULTS

In this section we explain the three cases of comparison and the two cases of pricing of two paradoxes. For each one we show that the maximal process is represented by equations 1' or 2'.

3.1 Common Ratio

The first paradox we solve is Common Ratio where the preference between two lotteries with high probabilities of gain $p_i \geq p'$, (i = 1,2) switches when we multiply the probabilities by K such that 0<K<1 where K is sufficiently small to obtain that the resulting lotteries now have low probabilities of gain $Kp_i < p'$, (i = 1,2) and the amounts remain constant (MacCrimon and Larsson, 1979; Hagen, 1979; Kagel et al., 1990)[6]. The first result is for the choice between two lotteries with high probabilities.

RESULT 3.1.1: *Let* (p_1,x_1) *be compared to* (p_2,x_2) *where* $p_i \in W \subset R \subset P$, $x_i \in X$, $i = 1,2$, *then the process* $J_{P\times X}\,(J_{W\times W}(p_1,p_2),\ J_{X\times X}\,(x_1,x_2))$ *is maximal.*

Proof: See Appendix.

The second result is for the case where the two probabilities in Result 3.1.1 are multiplied by a constant K.

RESULT 3.1.2: *Let* (Kp_1,x_1) *be compared to* (Kp_2,x_2) *where* $Kp_i \in L \subset R \subset P$, $x_i \in X$, $i = 1,2$, *then the process* $J_{P\times X}(J_{L\times L}(Kp_1,Kp_2),\ J_{X\times X}(x_1,x_2))$ *is maximal.*

Proof: See Appendix.

These results imply that the elements are first grouped by similarity and then the two probabilities are judged together. The proofs show that equation 1' corresponds to the maximal process of results 3.1.1 and 3.1.2. If the two p_i are on L rather than on W, by symmetry the individual judges them by considering the quality of L. The proofs also show that all other judgments than $J_{W\times W}(p_1,p_2)$ and $J_{L\times L}(p_1,p_2)$ are dominated. For example the judgment with R×R and P×P are possible but dominated.

Consequently, the common-ratio paradox is solved because the agent judges the p_i by considering the set W×W and the weight $\alpha_{W\times W} > 1$ of equation (1') in a first step and by considering L×L in a second step where $\alpha_{L\times L} < 1$. Consequently, if the two lotteries have the same expected value, the individual will necessarily change his ranking. In other words as in Rule 6 in MacCrimon and Larsson (1979) the individual weights the relative importance of probabilities versus monetary amounts depending on the fact that the p_i are higher or lower than p'. Example 1.2 illustrates this result.

3.2 Preference Reversal

The study of the preference-reversal paradox has to be decomposed in two parts because it involves both comparison and pricing of lotteries. We start with the comparison of lotteries.

3.2.1 Comparison of lotteries

Comparison of lotteries in the preference-reversal paradox indicates that most subjects choose the lottery with a high probability of gain when it is compared to a lottery with a low probability of gain (Tversky et al., 1990). The next result gives the maximal process for this case.

RESULT 3.2: *Let (p_1,x_1) be compared to (p_2,x_2) where $p_1 \in W \subset R \subset P$, $p_2 \in L \subset R \subset P$, $x_1,x_2 \in X$, then the process $J_{P\times X}(J_{W\times L}(p_1,p_2), J_{X\times X}(x_1,x_2))$ is maximal.*

Proof: See Appendix.

Consequently, if $p_1x_1 = p_2x_2$, the individual will choose the lottery (p_1,x_1). Again this result corresponds to equation (1'). One can note, however, that the pricing of each lottery (equation 2') is dominated by the comparison ($\tau_{S\times R} < \tau_{w\times L}$). The above result shows clearly that more weight is allowed to the judgment of probabilities than to monetary amounts as suggested by Slovic and Lichtenstein (1983). In our model this is due to the fact that the three different probabilities come from two different sets W and L. Example 1.1 illustrates this result. The cognitive process differs only in the two sets of judgments of the probabilities W×L, W×W, or L×L for the first three results above.

We now move to the pricing of lotteries. In the next result for the "judged-certainty-equivalent" there will be a bigger difference in the judgment process, because the individual uses the boundaries when judging the elements p_i and x_i.

3.2.2 "Judged-certainty-equivalent"

Here, the principal feature is that the lottery with a high probability of gain is underestimated when it is priced, and the lottery with a low probability of gain is overestimated (Birnbaum et al., 1992). The next result shows how these estimations come from a rational judgment of lotteries.

RESULT 3.3: *Let* (p_1,x_1) *be a lottery to be judged, then* $J_{P\times X}\,(J_{S\times R}(B_k,p_1),\, J_{X\times X}(0,x_1))$ *is maximal to obtain the "judged-certainty-equivalent" of the lottery.*

Proof: See Appendix.

Example 1.3 illustrates this result. The process in 3.3 is highly different to the one in 3.2. Here we find that both the overestimation and the underestimation depend on the difference between the two qualities “surety” and “risk”.

Moreover if we combine results 3.2 and 3.3 we solve completely the preference-reversal paradox. Consequently, this paradox is explained by the use of different frames, S and R for pricing and W and L for comparison.

4. CONCLUSION

The major conclusion of this article is that it is possible to develop a choice theory that explains the test results with the use of different restructuring operations (framing, grouping by similarity). These processes are optimal for a rational individual who has limited computational ability, which would be the most plausible characteristic for all individuals.

The two important characteristics of the model are that qualities are taken into account explicitly in the judgment function, and that the maximization of the degree of confidence implies the use of different processes. For the three results we find that the theory predicts that the sets considered for the judgment of probabilities are $S \times R$ for Result 3.3 (pricing of lotteries in preference reversal), $W \times L$ for Result 3.2 (comparison of lotteries in preference reversal) and $W \times W$ or $L \times L$ for Result 3.1 (common ratio). These variations explain the different results of each paradox. Of course there are alternative ways of using these two characteristics (degree of confidence, quality) and further investigations may be needed to find the best one (see footnotes).

One straight extension would be the introduction of the positive and negative qualities for monetary amounts proposed by Kahneman and Tversky (1979). The model may also be useful to explain the cases where $x_1,x_2 > 0$ for lotteries $(p_1,x_1;p_2,x_2)$. The use of the segregation concept (Kahneman and Tversky, 1979) suggests that the first movement from 0 to x_1 has the "surety" quality whereas the second movement from x_1 to x_2 has the "risk" quality. This way of proceeding may explain the choice and the pricing of these lotteries (Mellers et al., 1992, Birnbaum and Sutton, 1992).

FOOTNOTES

1. We assume that the utility function is linear with respect to x_i because the shape of the utility function is not important in solving the paradoxes in this paper. However, for paradoxes that involve large monetary amounts such as the St-Petersburgh paradox, it is evident that the concavity of u(x) plays a fundamental role.

2. The link between equations (1) and (2) is the following: The value of a lottery L_1 noted (p_1,x_1) that takes into account a lottery L_3 noted (p_3,x_3) can be represented by $V(L_1/L_3) = (p_3+(p_1-p_3))\cdot(x_3+(x_1-x_3))$. If L_3 is defined by using the boundaries B_k for probabilities and 0 for monetary amounts, we obtain equation 2. If $L_3 = (p_1+p_2/2,x_1+x_2/2)$ and if the comparison between L_1 and L_2 is represented by $V(L_1/L_3) - V(L_2/L_3)$, we obtain equation (1).

3. For $p<p'$ we have in equation (2), $w(p) = p+p'p-p^2 = 0$ when $p=0$, $w'(p) = 1+p'-2p > 0$ and $w''(p) = -2 < 0$. For $p \geq p'$ we have $w(p) = -pp'+p^2+p' = 1$ when $p=1$, $w'(p) = -p'+2p > 0$, and $w''(p) = 2 > 0$.

4. For our purpose we need only three axioms. However it is clear that other axioms will be necessary to directly eliminate comparisons that make no sense, such as the comparison between p_1 and x_2.

5.We can split the set into two sets, and again split each set into two other sets and so on. This way seems better to represent the grouping by similarity defined by Ranyard (1995), but for the cases in this paper the results are the same with both methods and the one we use is the simplest.

6. In the way we have defined the two qualities W and L, the common-ratio paradox is applied to two probabilities in W in a first step and then to two probabilities in L. Consequently, we do not cover the whole class of possible comparisons as the concept of subproportionality would permit. However, the other case that implies comparisons of probabilities in both W and L such as $(.9, x_1)$ vs $(.1, x_2)$ and $(.9K, x_1)$ vs $(.1K, x_2)$ with $K<1$ is considered later.

APPENDIX

Proof of 3.1.1: By Definition 4 we choose the judgment that has a maximal τ. Let us first consider that τ of $J_{W\times W}(p_1,p_2) \geq \tau$ of $J_{Q_\ell\times Q_j}(x_1,x_2)$ $\forall Q_\ell, Q_j$. $J_{Q_\ell\times Q_j}(p_1,p_2)$ can be done with the qualities of P,R,W by A1 and P×P, R×R, W×W are the three possible sets by A2. W×W is maximal by R1. The τ maximal for $J_{Q_\ell\times Q_j}(p_1,x_1)$, $J_{Q_\ell\times Q_j}(p_2,x_2)$, $J_{Q_\ell\times Q_j}(p_1,x_2)$ and $J_{Q_\ell\times Q_j}(p_2,x_1)$ is obtained with the set P×X by A1 and A2. These judgments are dominated by $J_{W\times W}(p_1,p_2)$ by R1 and R2. The judgments of one variable p_1, p_2, x_1 or x_2, where the boundaries of p_i are 0 and 1 and the boundary of x_i is 0 show that $\tau_{X\times X}$ is maximal for the judgments of x_i and $\tau_{S\times R}$ is maximal for p_i by R1, A1, A2. $J_{W\times W}(p_1,p_2)$ is maximal when it is compared with these judgments by R1 and the hypothesis that τ of $J_{W\times W}(p_1,p_2) \geq \tau$ of $J_{Q_\ell\times Q_j}(x_1,x_2)$.

We now have by Definition 4 a new set of three elements $\{x_1, x_2, J_{W\times W}(p_1,p_2)\}$ and we choose the judgment with the maximal τ. $\tau_{X\times X}$ is maximal for the judgment $J_{Q_\ell\times Q_j}(x_1,x_2)$ by A1. The maximal degree of confidence of $J_{Q_\ell\times Q_j}(x_i, J_{W\times W}(p_1,p_2))$ is $\tau_{P\times X}$ by A1 and A2 and it is dominated by $\tau_{X\times X}$ by R2. $J_{W\times W}(p_1,p_2)$ cannot be judged with a boundary by A3. For the judgment of x_1 or x_2 alone, $J_{X\times X}(x_1,x_2)$ is maximal because there is no refinement for X. If we now assume τ of $J_{W\times W}(p_1,p_2) < \tau$ of $J_{Q_\ell\times Q_j}(x_1,x_2)$ we obtain the same result.

Up to now we have two elements $J_{W\times W}(p_1,p_2)$ and $J_{X\times X}(x_1,x_2)$. The only possible judgment of these two elements is on P×X by A1, A2, and A3.

□

Proof of 3.1.2: As in the result 3.1.1, we have $J_{P\times X}(J_{Q_\ell\times Q_j}(Kp_1,Kp_2), J_{X\times X}(x_1,x_2))$, but here the judgment $(J_{Q_\ell\times Q_j}(Kp_1,Kp_2)$ with the qualities of W×W is impossible by A1, so L×L is maximal by R1, A1, and A2. □

Proof of 3.2: As in the result 3.1.1 we have $J_{P\times X}(J_{Q_\ell\times Q_j}(p_1,p_2), J_{X\times X}(x_1,x_2))$ but here the judgment $J_{Q_\ell\times Q_j}(p_1,p_2)$ with the quality of W×W is impossible by A1 so W×L is maximal by R1, A1, and A2.

□

Proof of 3.3: By Definition 4 we choose the judgment that has the maximal τ. Let τ of $J_{X\times X}(x_1,0) > \tau$ of $J_{S\times R}(B_k,p_1)$. The maximal τ for the judgment of x_1 and 0 is $\tau_{X\times X}$ by A1 and it is larger than the τ associated with the judgment $J_{Q_\ell\times Q_j}(p_1,x_1)$ by A1, A2, and R2. Any of the two judgment functions of p_1 and B_k considers the qualities of R and S by A1, A2, and R1 and this judgment is maximal for the set $\{p_1, J_2(x_1,0)\}$. If τ of $J_{X\times X}(x_1,0) \leq \tau$ of $J_{S\times R}(B_k,p_1)$ we obtain the same result.

For $\{J_{S\times R}(B_k,p_1),J_{X\times X}(x_1,0)\}$ the judgments of one of these two elements with boundaries do not exist by A3. $J_{P\times X}(J_{S\times R}(B_k,p_1),J_{X\times X}(x_1,0))$ is the maximal judgment by A1 and R2.

□

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