Valuing Credit Derivatives Using Gaussian Quadrature:
A Stochastic Volatility Framework

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Abstract

This paper proposes semi-closed-form solutions to value derivatives on mean reverting assets. A very general mean reverting process for the state variable and two stochastic volatility processes, the square-root process and the Ornstein-Uhlenbeck process, are considered. For both models, semi-closed-form solutions for characteristic functions are derived and then inverted using the Gauss-Laguerre quadrature rule so as to recover the cumulative probabilities. As benchmarks, European call options are valued within the following frameworks: Black and Scholes (1973) (represents constant volatility and no mean reversion), Longstaff and Schwartz (1995) (represents constant volatility and mean reversion), Heston (1993) and Zhu (2000) (represent stochastic volatility and no mean reversion). These comparisons show that numerical prices converge rapidly to the exact price. When applied to the general models proposed (represent stochastic volatility and mean reversion), the Gauss-Laguerre rule proves very efficient and very accurate. As applications, pricing formulas for credit spread options, caps, floors and swaps are derived. It is also shown that even weak mean reversion can have a major impact on option prices.

Keywords: Mean reversion, stochastic volatility, Gaussian quadrature, inverse Fourier transform, Feynman-Kac theorem, credit spread options, caps, floors, swaps.

JEL Classification: G13, C63.

Résumé

Cet article propose des solutions semi fermées pour évaluer des produits dérivés sur les actifs présentant un retour à la moyenne. On considère un processus très général avec retour à la moyenne pour la variable d'état ainsi que deux processus pour la volatilité stochastique: le processus racine carrée et le processus Ornstein-Uhlenbeck. Pour les deux modèles, on dérive des solutions semi fermées pour les fonctions caractéristiques et on calcule leurs transformées inverses de Fourier avec la quadrature Gauss-Laguerre afin de récupérer les probabilités cumulatives. Comme benchmarks, on évalue des options européennes d'achat dans les modèles suivants: Black et Scholes (1973) (représente une volatilité constante et aucun retour à la moyenne), Longstaff et Schwartz (1995) (représente une volatilité constante et un retour à la moyenne), Heston (1993) et Zhu (2000) (représentent une volatilité stochastique et aucun retour à la moyenne). Ces comparaisons permettent de voir que les prix semi analytiques convergent rapidement vers les prix exacts. Utilisée dans le cadre des nos modèles généraux (représentent une volatilité stochastique et un retour à la moyenne), la quadrature Gauss-Laguerre est très efficace et très précise. Comme applications directes, on propose des formules d'évaluation pour les options, les caps, les floors et les swaps sur écarts de crédit. On montre aussi que même un faible coefficient de retour à la moyenne peut avoir un grand impact sur les prix d'options.

Mots clés : Retour à la moyenne, volatilité stochastique, quadrature gaussienne, inversion de la transformée de Fourier, théorème de Feynman-Kac, options sur écarts de crédit, caps, floors, swaps.

Classification JEL : G13, C63.

Introduction

Several papers provide empirical evidence showing that mean reversion occurs naturally in interest rates and commodity markets and is therefore important in modeling and pricing derivatives. Longstaff and Schwartz (1995) show that log-credit spreads are mean reverting and derive a simple closed-form solution for European credit spread options. Schwartz (1997) also finds strong and significant mean reversion for oil and copper prices and proposes valuation formulas for some contingent claims. In more theoretical papers, Vasicek (1977) develops a continuous-time model of interest rates that incorporates mean reversion, while Cox, Ingersoll and Ross (1995) propose a general equilibrium model from which they derive the square-root interest rate diffusion,[1] the discount bond price as well as the bond option pricing formula. However, all these models assume a non-stochastic volatility, a strong simplification that can lead to unrealistic option prices.

Substantial progress has been made in developing more realistic option pricing models by incorporating stochastic volatility and jumps. Heston (1993) prices European options on stocks, bonds and currencies under a square-root volatility process. In the same way, Bakshi, Cao and Chen (1997) combine stochastic volatility and jumps to test the empirical performance of some alternative option pricing models. Schöbel and Zhu (1999) and Zhu (2000) propose a more elegant method to derive option prices under the square-root and the Ornstein-Uhlenbeck volatility models. While all these papers propose simple and easy-to-use closed-form solutions for derivatives on non-mean reverting assets under a stochastic volatility assumption, there is little literature for mean reverting assets within a stochastic volatility framework.

Assuming a square-root volatility and Vasicek's (1977) interest rate process, Fong and Vasicek (1992) develop the fundamental partial differential equation for interest rate contingent claims but derive a closed-form solution only for discount bonds. This solution requires a heavy computation of the confluent hypergeometric function within the complex numbers algebra. To override this difficulty, Selby and Strickland (1995) propose a series solution for the discount bond price that is very efficient. In addition, Clewlow and Strickland (1997) develop a Monte Carlo valuation of other interest rate derivatives under the Fong and Vasicek (1992) model.

In a discrete-time GARCH framework, Tahani (2000) confirms the empirical mean reversion for log-credit spreads reported by Longstaff and Schwartz (1995). Furthermore, he finds that the Heston and Nandi (2000) GARCH process fits the data better than the Gaussian process, and proposes a closed-form valuation formula for credit spread options. He also shows that the GARCH process used has the square-root mean reverting diffusion as a continuous-time limit.

This paper derives pricing formulas for options on mean reverting assets within two stochastic volatility frameworks: the square-root and the Ornstein-Uhlenbeck processes. We generalize the Longstaff and Schwartz (1995) constant volatility model, as well as the Heston (1993) and Zhu (2000) models by incorporating a mean reverting component. We also extend the Fong and Vasicek (1992) work by proposing a semi-analytic valuation framework for derivatives on general mean reverting assets, as an alternative to the Monte Carlo valuation presented by Clewlow and Strickland (1997). It is known that Monte Carlo methods require the simulation of a large number of paths, which makes semi-analytic valuation (when possible) much more efficient.

For both stochastic volatility models considered in this paper, we derive semi-closed-form characteristic functions for which simple ordinary differential equations (ODEs) must be solved. In the square-root case, even though a complete closed-form solution (involving the Whittaker functions linked to the hypergeometric functions) is derived, the numerical resolution of the ODEs provides very accurate solutions in much less time. In fact, when dealing with complex functions that can only be computed approximately as a series expansion, one is faced with time-consuming computation and frequent overflow errors even with mathematical softwares.

Once the characteristic function is derived in a semi-closed-form way, the inverse Fourier transform technique is applied to obtain the associated cumulative probabilities, using numerical integration based on the Gauss-Laguerre quadrature rule. The Gaussian integration technique has proved very efficient and very accurate in many papers including Bates (1996), who prices currency options within a stochastic volatility and jumps framework, and Sullivan (2000, 2001), who proposes an approximation to American options. In our frameworks, the Gauss-Laguerre quadrature rule is shown to be very accurate and convergent even with small polynomial degrees.

The contribution of this article is twofold. First, semi-analytic valuation formulas for derivatives on very general mean reverting underlying assets under a stochastic volatility assumption are proposed. As applications, solutions for credit spread options, caps, floors and swaps are derived. Second, it is shown that even weak mean reversion can have a large impact on option prices.

The next section presents a general mean reverting framework and shows how to compute the characteristic function. Section II derives semi-closed-form solutions for characteristic functions under both the square-root and the Ornstein-Uhlenbeck volatility assumptions. Section III presents the numerical integration procedure using Gaussian quadrature rules to recover the cumulative probabilities. Section IV derives valuation formulas for some credit spread derivatives as well as their Greeks as particular applications. Section V presents some empirical results on convergence and efficiency. Section VI will conclude.

I General mean reverting framework and characteristic functions

We consider a more general model for the state variable given under the historical measure P by:

$$dX_t = (\mu - \alpha X_t)dt + b(V_t)dZ_1(t) \qquad (1)$$

where X denotes any general mean reverting process such as log-credit spreads in Longstaff and Schwartz (1995) and Tahani (2000), or log-commodities in Schwartz (1997). For the volatility, we consider the following general diffusion:

$$d(a(V_t)) = \kappa(\theta - a(V_t))dt + b'(V_t)dZ_2(t) \qquad (2)$$

where $a(\cdot)$, $b(\cdot)$ and $b'(\cdot)$ are real-valued functions of the squared volatility V and will be specified later. The parameters μ, α, κ and θ are constant. Z_1 and Z_2 are two correlated Brownian motions under the historical measure P.

As in Heston (1993), the volatility risk-premium is taken proportional to $a(V)$ and the risk-premium for the state variable proportional to $b^2(V)$, such that the two diffusions under a risk-neutral measure Q become:

$$dX_t = [\mu - \alpha X_t - \gamma b^2(V_t)]dt + b(V_t)dW_1(t) \qquad (3)$$

$$d(a(V_t)) = [\kappa\theta - (\kappa + \pi)\, a(V_t)]dt + b'(V_t)dW_2(t) \tag{4}$$

where γ and π denote the unit risk-premiums and W_1 and W_2 are two correlated Brownian motions under Q. This modeling nests some special cases such as log-stock or log-currency diffusions (if $\alpha = 0$). To value option-like derivatives[2] with a time horizon T and a strike K, we must compute the following types of expectations under Q at time t:

$$E_t^Q\left(\exp\left(-\int_t^T r(s)ds\right) \times e^{X_T} \times 1_{X_T > \ln(K)}\right) \tag{5}$$

and

$$E_t^Q\left(\exp\left(-\int_t^T r(s)ds\right) \times 1_{X_T > \ln(K)}\right) \tag{6}$$

where E_t^Q denotes mathematical expectation taken under the probability measure Q conditioned on the information up to time t and r is the risk-free rate. In order to obtain simpler expressions for these expectations, we consider two probability measures Q_1 and Q_2 equivalent to Q and defined by their Radon-Nikodym derivatives:

$$\frac{dQ_1}{dQ} \equiv g_1(t,T) = \frac{\exp\left[-\int_t^T r(s)ds\right] \times e^{X_T}}{E_t^Q\left(\exp\left[-\int_t^T r(s)ds\right] \times e^{X_T}\right)} \tag{7}$$

$$\frac{dQ_2}{dQ} \equiv g_2(t,T) = \frac{\exp\left[-\int_t^T r(s)ds\right]}{E_t^Q\left(\exp\left[-\int_t^T r(s)ds\right]\right)} \tag{8}$$

Note that Q_2 is simply the so-called T-forward measure. Equations (5) and (7) give:

$$E_t^Q\left(\exp\left[-\int_t^T r(s)ds\right] \times e^{X_T} \times 1_{X_T > \ln(K)}\right) = Q_1^t(X_T > \ln(K)) \times E_t^Q\left(e^{-\int_t^T r(s)ds}\, e^{X_T}\right) \tag{9}$$

while Equations (6) and (8) give:

$$E_t^Q\left(\exp\left[-\int_t^T r(s)ds\right] \times 1_{X_T > \ln(K)}\right) = Q_2^t(X_T > \ln(K)) \times P(t,T) \tag{10}$$

where $Q_1^t(\cdot)$, $Q_2^t(\cdot)$ denote the probabilities conditioned on the information up to time t and $P(t,T)$ is the zero-coupon bond maturing at T. We also define the time-t conditioned characteristic functions of X_T under Q_1 and Q_2 by:

$$f_j(\phi) \equiv E_t^{Q_j}\left[\exp(i\phi X_T)\right] \qquad \text{for } j = 1, 2 \qquad (11)$$

Expressed under the risk-neutral measure Q, the characteristic function f_1 becomes:

$$
\begin{aligned}
f_1(\phi) &\equiv E_t^Q\left[g_1(t,T)\exp(i\phi X_T)\right] \\
&= \frac{E_t^Q\left[\exp\left(-\int_t^T r(s)ds\right) \times \exp((1+i\phi)X_T)\right]}{E_t^Q\left(\exp\left[-\int_t^T r(s)ds\right] \times e^{X_T}\right)}
\end{aligned}
\qquad (12)
$$

and f_2 becomes:

$$
\begin{aligned}
f_2(\phi) &\equiv E_t^Q\left[g_2(t,T)\exp(i\phi X_T)\right] \\
&= \frac{E_t^Q\left[\exp\left(-\int_t^T r(s)ds\right) \times \exp(i\phi X_T)\right]}{E_t^Q\left(\exp\left[-\int_t^T r(s)ds\right]\right)}
\end{aligned}
\qquad (13)
$$

These expressions will be derived dependently on the risk-free rate specification. However, if we define an "actualized characteristic function"[3] of X_T under Q by:

$$f(\psi) \equiv E_t^Q\left[\exp\left(-\int_t^T r(s)ds\right)\exp(\psi X_T)\right] \qquad (14)$$

we can simplify (12) and (13) as:

$$f_1(\phi) = \frac{f(1+i\phi)}{f(1)} \qquad (15)$$

$$f_2(\phi) = \frac{f(i\phi)}{f(0)} \qquad (16)$$

Under our stochastic volatility and mean reverting models, we will show that these characteristic functions can be expressed as log-linear combinations of some functions that solve simple ODEs.

To recover the cumulative probabilities in Equations (9) and (10), we apply the Fourier inversion transform (see Kendall and Stuart, 1977) to obtain:

$$Q_1(X_T > \ln(K)) = \frac{1}{2} + \frac{1}{\pi} \int_0^{+\infty} \text{Re}\left(\frac{f(1+i\phi)}{i\phi f(1)} K^{-i\phi} \right) d\phi \tag{17}$$

and

$$Q_2(X_T > \ln(K)) = \frac{1}{2} + \frac{1}{\pi} \int_0^{+\infty} \text{Re}\left(\frac{f(i\phi)}{i\phi f(0)} K^{-i\phi} \right) d\phi \tag{18}$$

where $\text{Re}(\cdot)$ denotes the real part of a complex number. It is straightforward to show that the integrands in Equations (17) and (18) are well defined for all $\phi \in [0, +\infty)$ and that the integrals are convergent (see Appendix C for details). Although they cannot be computed analytically, numerical techniques such as the Gaussian quadrature rule can be used to approximate these integrals.

The next section will consider two different stochastic volatility models by choosing appropriate $a(\cdot)$, $b(\cdot)$ and $b'(\cdot)$ functions according to Equations (3) and (4), and will derive the corresponding characteristic functions.

II Stochastic volatility models

II.1 Square-root mean reverting model

In this subsection, we generalize the Heston (1993) model by considering a mean reverting process X_t with a square-root stochastic volatility V_t. The model is given under the risk-neutral measure Q by:

$$dX_t = (\mu - \alpha X_t - \gamma V_t)dt + \sqrt{V_t} dW_1(t) \tag{19}$$

$$dV_t = (\kappa\theta - \lambda V_t)dt + \sigma\sqrt{V_t} dW_2(t) \tag{20}$$

where $d\langle W_1, W_2 \rangle_t = \rho dt$. For all models, we assume a constant risk-free rate denoted by r. The characteristic function can be expressed by (see Appendix A.1 for the derivation details):

$$E_t^Q\left(e^{\psi X_T}\right)=\exp\left(\begin{array}{c}\psi e^{-\alpha(T-t)}X_t+\dfrac{\mu}{\alpha}\psi\left(1-e^{-\alpha(T-t)}\right)-\dfrac{\rho}{\sigma}\dfrac{\kappa\theta}{\alpha}\psi\left(1-e^{-\alpha(T-t)}\right)\\[2mm]-\dfrac{\rho}{\sigma}\psi e^{-\alpha(T-t)}V_t\end{array}\right)$$

$$\times E_t^Q\left(\exp\left(\varepsilon_2 V_T-\int_t^T\varepsilon_1(T-s)V_s\,ds\right)\right)\tag{21}$$

where

$$\begin{cases}\varepsilon_1(\tau)=\left(\dfrac{\rho}{\sigma}(\alpha-\lambda)+\gamma\right)\psi\exp(-\alpha\tau)-\dfrac{1}{2}\psi^2\left(1-\rho^2\right)\exp(-2\alpha\tau)\\[3mm]\varepsilon_2=\dfrac{\rho}{\sigma}\psi\end{cases}\tag{22}$$

Putting $\alpha=0$ leads obviously to the corresponding equations in Zhu (2000). The expectation term in the right-hand side of Equation (21) will be computed using the Feynman-Kac theorem as given in Karatzas and Shreve (1991) (see Appendix D for details). Indeed, let $F(t,V)$ be defined by:

$$F(t,V)=E_{t,V}^Q\left[\exp\{\varepsilon_2 V_T\}\exp\left\{-\int_t^T\varepsilon_1(T-s)V_s\,ds\right\}\right]\tag{23}$$

then $F(t,V)$ must satisfy the following partial differential equation (PDE):

$$\begin{cases}\dfrac{\partial F}{\partial t}+\dfrac{1}{2}\sigma^2 V\dfrac{\partial^2 F}{\partial V^2}+(\kappa\theta-\lambda V)\dfrac{\partial F}{\partial V}-\varepsilon_1(T-t)V\,F=0\\[4mm]F(T,V)=\exp(\varepsilon_2 V)\end{cases}\tag{24}$$

Assuming that $F(t,V)$ is log-linear and given by:

$$F(t,V)=\exp\left[D(T-t)V+C(T-t)\right]\tag{25}$$

yields that $D(\cdot)$ and $C(\cdot)$, expressed as functions of the time variable $\tau=T-t$, must satisfy the following ODEs:

$$\begin{cases}D'(\tau)-\dfrac{1}{2}\sigma^2 D^2(\tau)+\lambda D(\tau)+\varepsilon_1(\tau)=0\\[3mm]D(0)=\varepsilon_2=\dfrac{\rho}{\sigma}\psi\end{cases}\tag{26}$$

and

$$\begin{cases} C'(\tau) - \kappa\theta\, D(\tau) = 0 \\[2mm] C(0) = 0 \end{cases} \tag{27}$$

Solving these ODEs will give the unique solution to the PDE (24) and then to the actualized characteristic function $f(\psi)$. Although the ODE (26) is of Riccati type, there is no simple analytic solution as in Heston (1993) and Zhu (2000)[4] (see Appendix A.2 for details). Because of the mean reverting feature, the function $\varepsilon_1(\tau)$ in Equation (22) is of an exponential type, whereas it is a constant parameter independent of the time variable τ in Heston (1993) and Zhu (2000). However, these first-degree ODEs can be solved easily using numerical methods such as the Runge-Kutta formula or the Adams-Bashforth-Moulton method. For details about these methods, refer to Dormand and Prince (1980), Shampine (1994) or Shampine and Gordon (1975).

The actualized characteristic function $f(\psi)$ for the square-root mean reverting model is then given by:

$$f(\psi) \equiv E_t^Q\left[\exp\left(-\int_t^T r(s)ds\right)\exp(\psi X_T)\right]$$

$$= e^{-r(T-t)} \times \exp\left(\begin{array}{c} \psi e^{-\alpha(T-t)} X_t + \dfrac{\mu}{\alpha}\psi\left(1 - e^{-\alpha(T-t)}\right) - \dfrac{\rho}{\sigma}\dfrac{\kappa\theta}{\alpha}\psi\left(1 - e^{-\alpha(T-t)}\right) \\[3mm] -\dfrac{\rho}{\sigma}\psi e^{-\alpha(T-t)} V_t \end{array}\right) \tag{28}$$

$$\times \exp\left[D(T-t;\psi)V_t + C(T-t;\psi)\right]$$

Now that we can evaluate numerically expressions like $f(i\phi)$ and $f(1+i\phi)$ for all $\phi \in [0,+\infty)$, we are able to compute cumulative probabilities under measures Q_1 and Q_2 by inverting their Fourier transforms. We use the Gaussian quadrature rule with Laguerre polynomials so as to evaluate the integrals given in Equations (17) and (18). Section III will describe this method and show how it applies to our models.

II.2 Ornstein-Uhlenbeck mean reverting model

A. Risk-premium for the state variable is proportional to the squared volatility

In this subsection, we use an Ornstein-Uhlenbeck model for the volatility σ_t and a mean reverting process X_t. The risk-premium is assumed to be proportional to the squared volatility σ_t^2. The model is then given under the risk-neutral measure Q by:

$$dX_t = \left(\mu - \alpha X_t - \gamma \sigma_t^2\right)dt + \sigma_t dW_1(t) \tag{29}$$

$$d\sigma_t = \left(\kappa\theta - \lambda\sigma_t\right)dt + \beta\, dW_2(t) \tag{30}$$

where $d\langle W_1, W_2 \rangle_t = \rho dt$. The characteristic function is given by (see Appendix B.1 for the derivation details):

$$E_t^Q\left(e^{\psi X_T}\right) = \exp\left(\begin{array}{c} \psi e^{-\alpha(T-t)} X_t + \dfrac{\mu}{\alpha}\psi\left(1 - e^{-\alpha(T-t)}\right) - \dfrac{\rho\beta}{2\alpha}\psi\left(1 - e^{-\alpha(T-t)}\right) \\ -\dfrac{\rho}{2\beta}\psi e^{-\alpha(T-t)}\sigma_t^2 \end{array}\right) \\ \times E_t^Q\left(\exp\left(\eta_3\sigma_T^2 - \int_t^T \eta_2(T-s)\sigma_s ds - \int_t^T \eta_1(T-s)\sigma_s^2 ds\right)\right) \tag{31}$$

where

$$\begin{cases} \eta_1(\tau) = \left(\dfrac{\alpha\rho}{2\beta} - \dfrac{\rho\lambda}{\beta} + \gamma\right)\psi\exp(-\alpha\tau) - \dfrac{1}{2}\psi^2\left(1 - \rho^2\right)\exp(-2\alpha\tau) \\ \eta_2(\tau) = \dfrac{\rho\kappa\theta}{\beta}\psi\exp(-\alpha\tau) \\ \eta_3 = \dfrac{\rho}{2\beta}\psi \end{cases} \tag{32}$$

Again, putting $\alpha = 0$ leads to the corresponding equations in Zhu (2000). As before, let $G(t,\sigma)$ be defined by:

$$G(t,\sigma) = E_t^Q\left(\exp\left(\eta_3\sigma_T^2 - \int_t^T \eta_2(T-s)\sigma_s ds - \int_t^T \eta_1(T-s)\sigma_s^2 ds\right)\right) \tag{33}$$

Then according to the Feynman-Kac theorem, $G(t,\sigma)$ must satisfy the following PDE:

$$\begin{cases} \dfrac{\partial G}{\partial t} + \dfrac{1}{2}\beta^2 \dfrac{\partial^2 G}{\partial \sigma^2} + (\kappa\theta - \lambda\sigma)\dfrac{\partial G}{\partial \sigma} - \left(\eta_1(T-t)\sigma^2 + \eta_2(T-t)\sigma\right)G = 0 \\[2mm] G(T,\sigma) = \exp\left(\eta_3\sigma^2\right) \end{cases} \qquad (34)$$

Assuming that $G(t,\sigma)$ is log-linear and given by:

$$G(t,\sigma) = \exp\left[\dfrac{1}{2}E(T-t)\sigma^2 + D(T-t)\sigma + C(T-t)\right] \qquad (35)$$

yields that $E(\cdot)$, $D(\cdot)$ and $C(\cdot)$ expressed as functions of the time variable $\tau = T - t$ must satisfy the following ODEs:

$$\begin{cases} \dfrac{1}{2}E'(\tau) - \dfrac{1}{2}\beta^2 E^2(\tau) + \lambda E(\tau) + \eta_1(\tau) = 0 \\[2mm] E(0) = 2\eta_3 = \dfrac{\rho}{\beta}\psi \end{cases} \qquad (36)$$

$$\begin{cases} D'(\tau) - \beta^2 E(\tau)D(\tau) + \lambda D(\tau) - \kappa\theta E(\tau) + \eta_2(\tau) = 0 \\[2mm] D(0) = 0 \end{cases} \qquad (37)$$

and

$$\begin{cases} C'(\tau) - \dfrac{1}{2}\beta^2 E(\tau) - \dfrac{1}{2}\beta^2 D^2(\tau) - \kappa\theta D(\tau) = 0 \\[2mm] C(0) = 0 \end{cases} \qquad (38)$$

Although the Riccati-type ODE (36) has an exact analytic solution (see Appendix B.2 for details), the ODEs (37) and (38) don't have closed-form solutions. As discussed earlier, all these ODEs will be solved numerically. The actualized characteristic function $f(\psi)$ for the Ornstein-Uhlenbeck mean reverting model is then given by:

$$f(\psi) \equiv E_t^Q \left[\exp\left(-\int_t^T r(s)ds \right) \exp(\psi X_T) \right]$$

$$= e^{-r(T-t)} \exp\left(\begin{array}{c} \psi e^{-\alpha(T-t)} X_t + \dfrac{\mu}{\alpha}\psi\left(1 - e^{-\alpha(T-t)}\right) - \dfrac{\rho\beta}{2\alpha}\psi\left(1 - e^{-\alpha(T-t)}\right) \\ -\dfrac{\rho}{2\beta}\psi e^{-\alpha(T-t)}\sigma_t^2 \end{array} \right) \tag{39}$$

$$\times \exp\left[\frac{1}{2}E(T-t;\psi)\sigma_t^2 + D(T-t;\psi)\sigma_t + C(T-t;\psi) \right]$$

B. Risk-premium for the state variable is proportional to the volatility

If the risk-premium for the state variable is proportional to the volatility instead of its square, then the model's equations become:

$$dX_t = \left(\mu - \alpha X_t - \gamma\sigma_t\right)dt + \sigma_t dW_1(t) \tag{40}$$

$$d\sigma_t = \left(\kappa\theta - \lambda\sigma_t\right)dt + \beta\, dW_2(t) \tag{41}$$

With the same calculations as before, the characteristic function is obtained as:

$$E_t^Q\left(e^{\psi X_T}\right) = \exp\left(\begin{array}{c} \psi e^{-\alpha(T-t)} X_t + \dfrac{\mu}{\alpha}\psi\left(1 - e^{-\alpha(T-t)}\right) - \dfrac{\rho\beta}{2\alpha}\psi\left(1 - e^{-\alpha(T-t)}\right) \\ -\dfrac{\rho}{2\beta}\psi e^{-\alpha(T-t)}\sigma_t^2 \end{array} \right)$$

$$\times E_t^Q\left(\exp\left(\omega_3\sigma_T^2 - \int_t^T \omega_2(T-s)\sigma_s ds - \int_t^T \omega_1(T-s)\sigma_s^2 ds \right) \right) \tag{42}$$

where

$$\begin{cases} \omega_1(\tau) = \dfrac{\rho}{\beta}\left(\dfrac{\alpha}{2} - \lambda\right)\psi\exp(-\alpha\tau) - \dfrac{1}{2}\psi^2\left(1 - \rho^2\right)\exp(-2\alpha\tau) \\[2mm] \omega_2(\tau) = \left(\dfrac{\rho\kappa\theta}{\beta} + \gamma\right)\psi\exp(-\alpha\tau) \\[2mm] \omega_3 = \dfrac{\rho}{2\beta}\psi \end{cases} \tag{43}$$

Using the Feynman-Kac theorem gives for the actualized characteristic function:

$$f(\psi) \equiv E_t^Q \left[\exp\left(-\int_t^T r(s)ds \right) \exp(\psi X_T) \right]$$

$$= e^{-r(T-t)} \exp\left(\begin{array}{c} \psi e^{-\alpha(T-t)} X_t + \dfrac{\mu}{\alpha}\psi\left(1 - e^{-\alpha(T-t)}\right) - \dfrac{\rho\beta}{2\alpha}\psi\left(1 - e^{-\alpha(T-t)}\right) \\[2mm] -\dfrac{\rho}{2\beta}\psi e^{-\alpha(T-t)}\sigma_t^{\,2} \end{array} \right) \tag{44}$$

$$\times \exp\left[\frac{1}{2} E(T-t;\psi)\sigma_t^{\,2} + D(T-t;\psi)\sigma_t + C(T-t;\psi) \right]$$

where $E(\cdot)$, $D(\cdot)$ and $C(\cdot)$ solve the same ODEs (36)-(38) by replacing $\eta_1(\cdot), \eta_2(\cdot)$ and η_3 respectively by $\omega_1(\cdot), \omega_2(\cdot)$ and ω_3.

III   Numerical integration using Gaussian quadrature

In general, a quadrature rule allows to approximate an integral of a weighted function, $g(\cdot)$, over a given interval $[a,b]$ with a linear combination of function values in the same interval. After specifying a set of abscissas $(\phi_j)_{j=1,\dots,n}$ and their corresponding weights $(\omega_j)_{j=1,\dots,n}$, the integral is then approximated by:

$$\int_a^b w(\phi)g(\phi)d\phi \cong \sum_{j=1}^n \omega_j g(\phi_j) \tag{45}$$

where $w(\cdot)$ is a weight function to be specified depending on the rule used. The abscissas and the weights are specified such that this approximation is exact for any given polynomial function with a maximum degree. The highest degree is called the order of the quadrature rule. While rules such as the trapezoidal and the Simpson's specify a set of equally spaced abscissas and choose the weights to maximize the order, Gaussian rules determine both abscissas and weights to maximize the order. For n abscissas and n weights, the highest order is $2n-1$. Furthermore, in many studies, Gaussian rules are shown to converge faster than the classic trapezoidal and Simpson's rules and give greater accuracy even for small n (see Sullivan, 2000 and 2001).

The Gauss-Laguerre quadrature rule over the interval $[0,+\infty)$ has the following weight function $w(\phi) = \exp(-\phi)$. The abscissas and the weights solve the following $2n$ equations:

$$\int_{0}^{+\infty} \exp(-\phi)\phi^{q}\,d\phi = \omega_{1}\exp(-\phi_{1})\phi_{1}^{q} + \omega_{2}\exp(-\phi_{2})\phi_{2}^{q} + \ldots + \omega_{n}\exp(-\phi_{n})\phi_{n}^{q} \quad (46)$$

for $q = 0, \cdots, 2n-1$. These abscissas and weights can also be determined using some properties of Laguerre polynomials. They are tabulated in Abramowitz and Stegun (1968). The next subsection gives a brief overview of these polynomials and shows how to specify the rule. The one after will apply this quadrature rule to invert the characteristic functions to recover cumulative probabilities.

III.1 A brief overview of Laguerre polynomials

The n-th Laguerre polynomial is defined by:

$$L_{n}(\phi) \equiv \frac{1}{n!}\exp(\phi) \times \frac{d^{n}}{d\phi^{n}}\left[\exp(-\phi)\phi^{n}\right]$$

$$= \frac{(-1)^{n}}{n!}\phi^{n} + \ldots + \frac{1}{2!}\phi^{2} - \phi + 1 \quad (47)$$

These polynomials have many characteristics, among which their "orthonormality" with respect to the weight function:

$$\int_{0}^{+\infty} \exp(-\phi)L_{n}(\phi)L_{p}(\phi)\,d\phi = \delta_{np} = \begin{cases} 0 & \text{if } n \neq p \\ 1 & \text{if } n = p \end{cases} \quad (48)$$

where δ_{np} is the Kronecker's symbol. It is also known that the n-th Lageurre polynomial has exactly n real zeros over the interval $(0, +\infty)$. These zeros are the abscissas $\left(\phi_{j}\right)_{j=1,\ldots,n}$ needed for the Gauss-Laguerre quadrature rule of order n. The associated weights $\left(\omega_{j}\right)_{j=1,\ldots,n}$ are given by:

$$\omega_{j} = \frac{1}{n^{2}}\frac{\phi_{j}}{\left(L_{n-1}(\phi_{j})\right)^{2}} \qquad j = 1,\ldots,n \quad (49)$$

In order to apply this integration method to our models, we need to modify the weights to take into account the function to be integrated. As seen in the previous sections, we have to value this type of integral:

$$\int\limits_0^{+\infty} g(\phi)d\phi = \int\limits_0^{+\infty} \exp(-\phi)[\exp(\phi)g(\phi)]d\phi$$

$$\cong \sum_{j=1}^n \omega_j \exp(\phi_j)g(\phi_j)$$

(50)

When n increases, the sum converges to the true value of the integral if the function $\exp(\phi)g(\phi)$ satisfies certain assumptions, as discussed in Davis and Rabinowitz (1984).

III.2 Recovering cumulative probabilities

For a given order n, we set up the abscissas $(\phi_j)_{j=1,...,n}$ and the modified weights $(\omega_j \exp(\phi_j))_{j=1,...,n}$ as shown before. All the model's parameters must be fixed, as well as the time t, the maturity T, the strike K, the initial underlying asset log-value X_t and the initial value of the volatility σ_t or the squared volatility V_t depending on the model. For every abscissa ϕ_j or equivalently ψ_j as defined earlier by $\psi_j = 1 + i\phi_j$ or $\psi_j = i\phi_j$, we solve the ODEs (26) and (27) for the square-root model and Equations (36)-(38) for the Ornstein-Uhlenbeck model to get the function values $E(T-t;\psi_j)$, $D(T-t;\psi_j)$ and $C(T-t;\psi_j)$ defined in Section II for every $j=1,...,n$. The actualized characteristic function values $f(\psi_j)$ can then be computed at the needed points ψ_j and the cumulative probabilities can be approximated by:

$$Q_1(X_T > \ln(K)) = \frac{1}{2} + \frac{1}{\pi}\int\limits_0^{+\infty} \mathrm{Re}\left(\frac{f(1+i\phi)}{i\phi f(1)}K^{-i\phi}\right)d\phi$$

$$\cong \frac{1}{2} + \frac{1}{\pi}\sum_{j=1}^n \omega_j \exp(\phi_j)\,\mathrm{Re}\left(\frac{f(1+i\phi_j)}{i\phi_j f(1)}\exp\{-i\phi_j \ln(K)\}\right)$$

(51)

and

$$Q_2(X_T > \ln(K)) = \frac{1}{2} + \frac{1}{\pi}\int\limits_0^{+\infty} \mathrm{Re}\left(\frac{f(i\phi)}{i\phi f(0)}K^{-i\phi}\right)d\phi$$

$$\cong \frac{1}{2} + \frac{1}{\pi}\sum_{j=1}^n \omega_j \exp(\phi_j)\,\mathrm{Re}\left(\frac{f(i\phi_j)}{i\phi_j f(0)}\exp\{-i\phi_j \ln(K)\}\right)$$

(52)

Theoretically, as *n* becomes large, these approximations converge to the true probability values. As it will be shown with many valuation examples, a fast convergence and a good accuracy can be achieved even with small *n*.

IV Credit spread option, cap, floor and swap valuation

IV.1 Credit spread option

A credit spread option gives the right to buy or sell the credit at the strike price until or at the expiration date depending on whether the option is American or European. One could buy a credit spread option for hedging its credit risk exposures against up or down movements in a credit value as well as for speculative purposes. For an exhaustive credit derivatives overview, see Howard (1995).

More specifically, denoting the maturity date by *T* and the strike by *K*, under the models studied in earlier sections, the European call premium is given by:

$$Call(t,T) = f(t,T;1)\, Q_1^{t,T}\left(X_T > \ln(K)\right) - f(t,T;0)\, K\, Q_2^{t,T}\left(X_T > \ln(K)\right) \qquad (53)$$

where the actualized characteristic function $f(t,T;\psi)$ and the cumulative probabilities are defined as before by:

$$f(t,T;\psi) = E_t^Q\left[\exp\left(-\int_t^T r(s)ds\right)\exp(\psi X_T)\right] \qquad (54)$$

and

$$Q_1^{t,T}\left(X_T > \ln(K)\right) = \frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty}\mathrm{Re}\left(\frac{f(t,T;1+i\phi)}{i\phi f(t,T;1)}K^{-i\phi}\right)d\phi \qquad (55)$$

$$Q_2^{t,T}\left(X_T > \ln(K)\right) = \frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty}\mathrm{Re}\left(\frac{f(t,T;i\phi)}{i\phi f(t,T;0)}K^{-i\phi}\right)d\phi \qquad (56)$$

The European put can be derived using the call-put parity:

$$Put(t,T) = f(t,T;0)\, K\, Q_2^{t,T}\left(X_T < \ln(K)\right) - f(t,T;1)\, Q_1^{t,T}\left(X_T < \ln(K)\right) \qquad (57)$$

In order to hedge options against changes in the underlying asset and in the volatility, the sensitivities (i.e. the *Greeks*) must be derived (see Appendix E for details). For both stochastic

volatility models presented in Section II, a straightforward calculation shows that the *Delta* is given by:

$$Delta(t,T) \equiv \frac{\partial Call(t,T)}{\partial e^{X_t}} = e^{-\alpha(T-t)} e^{-X_t} f(t,T;1) Q_1^{t,T}(X_T > \ln(K)) \tag{58}$$

and the *Gamma* by:

$$Gamma(t,T) \equiv \frac{\partial Delta(t,T)}{\partial e^{X_t}}$$

$$= \left(e^{-\alpha(T-t)} - 1\right) e^{-X_t} Delta(t,T) \tag{59}$$

$$+ \frac{e^{-2\alpha(T-t)} e^{-2X_t}}{\pi} \int_0^{+\infty} \text{Re}\left(f(t,T;1+i\phi)K^{-i\phi}\right)d\phi$$

The derivation of the *Vega* depends on the stochastic volatility model that is used. For the square-root model, the *Vega* is given by:

$$Vega(t,T) \equiv \frac{\partial Call(t,T)}{\partial V_t}$$

$$= \left(D(T-t;1) - \frac{\rho}{\sigma}e^{-\alpha(T-t)}\right) f(t,T;1) Q_1^{t,T}(X_T > \ln(K))$$

$$+ \frac{1}{\pi}\int_0^{+\infty}\text{Re}\left(\frac{D(T-t;1+i\phi) - D(T-t;1)}{i\phi} f(t,T;1+i\phi)K^{-i\phi}\right)d\phi \tag{60}$$

$$- \frac{K}{\pi}\int_0^{+\infty}\text{Re}\left(\frac{D(T-t;i\phi)}{i\phi} f(t,T;i\phi)K^{-i\phi}\right)d\phi$$

while for the Ornstein-Uhlenbeck model, it is expressed as:

$$Vega(t,T) \equiv \frac{\partial Call(t,T)}{\partial \sigma_t}$$

$$= \left(E(T-t;1)\sigma_t + D(T-t;1) - \frac{\rho}{\beta}e^{-\alpha(T-t)}\sigma_t\right) f(t,T;1) Q_1^{t,T}(X_T > \ln(K))$$

$$+ \frac{1}{\pi}\int_0^{+\infty}\text{Re}\left(\left[\frac{\frac{E(T-t;1+i\phi) - E(T-t;1)}{i\phi}\sigma_t}{+\frac{D(T-t;1+i\phi) - D(T-t;1)}{i\phi}}\right] f(t,T;1+i\phi)K^{-i\phi}\right)d\phi \tag{61}$$

$$- \frac{K}{\pi}\int_0^{+\infty}\text{Re}\left(\left[\frac{E(T-t;i\phi)}{i\phi}\sigma_t + \frac{D(T-t;i\phi)}{i\phi}\right] f(t,T;i\phi)K^{-i\phi}\right)d\phi$$

IV.2 Credit spread cap and floor

A credit spread cap or floor provides the right to get payoffs at periodic dates called the *reset dates*. At each reset date, the cap/floor payoff is the same as for a call/put. In this, the cap/floor can be seen as a sequence of many calls/puts called *caplets* or *floorlets*. The illustration below shows the reset dates and the associated payoffs for a credit spread cap with a maturity T and different strike K_j corresponding to the n periods.
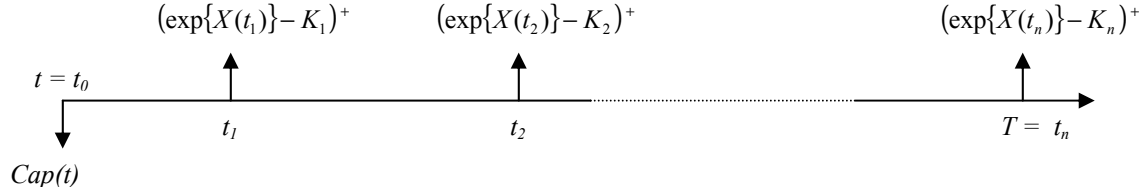
The cap/floor premium is then equal to the sum of the corresponding caplets/floorlets premia. The cap premium is given by:

$$Cap(t) = \sum_{j=1}^{n} f(t,t_j;1) Q_1^j \left(X(t_j) > \ln(K_j) \right)$$
$$- \sum_{j=1}^{n} f(t,t_j;0) K_j Q_2^j \left(X(t_j) > \ln(K_j) \right) \tag{62}$$

where $f(t,t_j;\psi)$, $Q_1^j \left(X(t_j) > \ln(K_j) \right)$ and $Q_2^j \left(X(t_j) > \ln(K_j) \right)$ are defined for each reset date as for the call in Equations (54)-(56). The floor premium can also be valued by:

$$Floor(t) = \sum_{j=1}^{n} f(t,t_j;0) K_j Q_2^j \left(X(t_j) < \ln(K_j) \right)$$
$$- \sum_{j=1}^{n} f(t,t_j;1) Q_1^j \left(X(t_j) < \ln(K_j) \right) \tag{63}$$

as well as the *Delta* by:

$$\frac{\partial Cap(t)}{\partial e^{X_t}} = e^{-X_t} \times \sum_{j=1}^{n} e^{-\alpha(t_j - t)} f(t,t_j;1) Q_1^j \left(X(t_j) > \ln(K_j) \right) \tag{64}$$

IV.3 Credit spread swap

A credit spread swap is an obligation to get payoffs at periodic reset dates. At each reset date, the swap payoff is the same as for a forward contract. In this, the swap can be seen as a sequence of many forward contracts called *swaplets*. The illustration below shows

the reset dates and the associated payoffs for a credit spread swap with maturity T and a constant strike K.
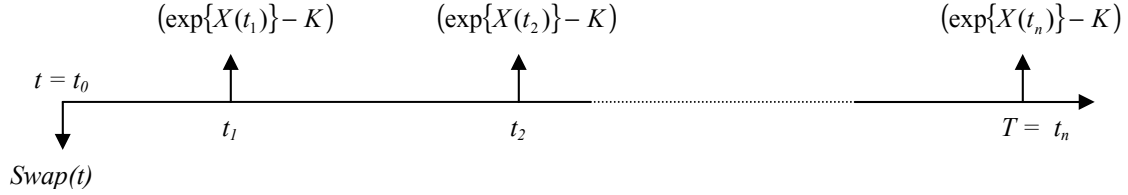
The swap value is then equal to the difference between the cap and the floor values with the same strike at the reset dates. By analogy with interest rate swaps, we can derive the value of the strike that makes the swap value equal to 0 at time t:

$$K = \frac{\sum_{j=1}^{n} f(t,t_j;1)}{\sum_{j=1}^{n} f(t,t_j;0)} \tag{65}$$

where $f(t,t_j;\psi)$ is defined for each reset date as in Equation (54).

V Empirical results on convergence and the impact of mean reversion

To assess the accuracy and the efficiency of our procedure, we price many European call options within different frameworks: Black and Scholes (1973) (B&S hereafter), Longstaff and Schwartz (1995) (L&S hereafter) and Zhu (2000).[5]

For different parameters settings, we try to converge to the exact price given by the corresponding simple analytic formulas for the B&S and L&S frameworks and by the *Matlab*[®] numerical integration routine for the Zhu framework. Tables 1 to 4 present the call prices for different maturities and quadrature orders. It is shown that a good accuracy is achieved even with small rule orders, between 10 and 15 depending on which model is used and within which framework.

[Insert tables 1 to 4 here]

We also compute the relative pricing errors between the exact call price and that obtained by the semi-analytic procedure. Figures 1 to 4 show that these errors are very small

and converge rapidly to 0. Notice that, in the Zhu (2000) framework, the square-root model converges faster than the Ornstein-Uhlenbeck model.

[Insert Figures 1 to 4 here]

The efficiency and the accuracy of our semi-analytic procedure within these exact frameworks are still true for the mean reverting framework. Indeed, for both the square-root and the Ornstein-Uhlenbeck mean reverting models, the "true" asymptotic price [6] is attained even with small rule orders, between 12 and 15. Tables 5 and 6 present these results on the convergence.

[Insert Tables 5 and 6 here]

The relative pricing errors in this case are computed with respect to the asymptotic price. Figures 5 and 6 show that these errors tend rapidly to 0. Again, the convergence is faster for the square-root mean reverting model than for the Ornstein-Uhlenbeck mean reverting model.

[Insert Figures 5 and 6 here]

It is also found that even weak mean reversion can have a major impact on option prices. The results presented in Tables 7 and 8 show that for a small mean reversion coefficient,   between 0.01 and 0.03, the relative difference with respect to the "no mean reversion option price" (i.e. $\alpha = 0$) ranges from 4% to 66%, depending on the moneyness and the maturity, which is substantial.

[Insert Tables 7 and 8 here]

VI Conclusion

In this paper, we propose semi-analytic pricing formulas for derivatives on mean reverting assets within two stochastic volatility frameworks. In this, we generalize Longstaff

and Schwartz (1995) by making the volatility stochastic, and Heston (1993) and Zhu (2000) by incorporating a mean reverting component in the underlying asset diffusion. This work also extends the Fong and Vasicek (1992) model, since a semi-analytic valuation of options on general mean reverting assets is proposed instead of a Monte Carlo simulation.

However, adding a mean reverting component to our models only allows access to semi-closed-form characteristic functions, in the sense that we need to solve some ODEs with numerical methods such as Runge-Kutta or Adams-Bashforth-Moulton. The numerical resolution is very accurate and takes much less time than the exact computation, since analytic solutions (when they exist) involve complex algebra with Whittaker functions and hypergeometric functions. A numerical integration method, such as the Gaussian quadrature, is needed to invert the characteristic function so as to recover the cumulative probabilities, and then to price derivatives. In our case, the Gauss-Laguerre rule proves very efficient and very accurate. As particular applications to this general valuation framework, we derive semi-closed-form pricing formulas for credit-spread European options, caps, floors, and swaps, as well as their *Greeks*.

We also find that the impact of even weak mean reversion on option prices could be very large. This finding proves that the pricing of derivatives on mean reverting underlying assets is very sensitive to the strength of the reversion, which therefore has to be taken into account in financial modeling.

The combination of numerical resolution of ODEs with numerical integration using Gaussian quadrature rules provides extremely accurate and efficient valuation of credit derivatives and thus may do well for derivatives on other mean reverting assets like interest rates and commodities.

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Tables

Table 1: Convergence to the Black-Scholes call price

Maturity	3 months	6 months	9 months	1 year
B&S price	4,614997	6,888729	8,772268	10,450584
n				
7	4,576689	6,888989	8,772230	10,450610
8	4,605435	6,888729	8,772276	10,450578
9	4,613204	6,888726	8,772267	10,450584
10	4,614770	6,888729	8,772268	10,450583
11	4,614983	6,888729	8,772268	10,450584
12	4,614998	6,888729	8,772268	10,450584
13	4,614997	6,888729	8,772268	10,450584
14	4,614997	6,888729	8,772268	10,450584
15	4,614997	6,888729	8,772268	10,450584
16	4,614997	6,888729	8,772268	10,450584
17	4,614997	6,888729	8,772268	10,450584
18	4,614997	6,888729	8,772268	10,450584
19	4,614997	6,888729	8,772268	10,450584
20	4,614997	6,888729	8,772268	10,450584
21	4,614997	6,888729	8,772268	10,450584
22	4,614997	6,888729	8,772268	10,450584
23	4,614997	6,888729	8,772268	10,450584
24	4,614997	6,888729	8,772268	10,450584
25	4,614997	6,888729	8,772268	10,450584

Table 1 pertains to the setup in Section II.1 and presents the results of the valuation of an at-the-money call within the Black and Scholes (1973) (B&S) framework for different quadrature rule orders n. The true price is given by the B&S analytic formula. The option's parameters are $X = \ln(100)$; $K = 100$; $r = 0.05$ and $V = 0.04$. To match the B&S framework, the model's parameters are $\mu = r$; $\alpha = 0$; $\gamma = 0.5$; $\rho = 0$; $\sigma = 0$; $\lambda = 0$; $\kappa = 0$ and $\theta = 0$.

Table 2: Convergence to the Longstaff-Schwartz call price

Maturity	3 months	6 months	9 months	1 year
L&S price	1,066132E-03	1,681529E-03	2,229959E-03	2,746019E-03
n				
7	1,059709E-03	1,681586E-03	2,229965E-03	2,746012E-03
8	1,064684E-03	1,681522E-03	2,229960E-03	2,746019E-03
9	1,065903E-03	1,681529E-03	2,229959E-03	2,746018E-03
10	1,066112E-03	1,681529E-03	2,229959E-03	2,746019E-03
11	1,066132E-03	1,681529E-03	2,229959E-03	2,746019E-03
12	1,066132E-03	1,681529E-03	2,229959E-03	2,746019E-03
13	1,066132E-03	1,681529E-03	2,229959E-03	2,746019E-03
14	1,066132E-03	1,681529E-03	2,229959E-03	2,746019E-03
15	1,066132E-03	1,681529E-03	2,229959E-03	2,746019E-03
16	1,066132E-03	1,681529E-03	2,229959E-03	2,746019E-03
17	1,066132E-03	1,681529E-03	2,229959E-03	2,746019E-03
18	1,066132E-03	1,681529E-03	2,229959E-03	2,746019E-03
19	1,066132E-03	1,681529E-03	2,229959E-03	2,746019E-03
20	1,066132E-03	1,681529E-03	2,229959E-03	2,746019E-03
21	1,066132E-03	1,681529E-03	2,229959E-03	2,746019E-03
22	1,066132E-03	1,681529E-03	2,229959E-03	2,746019E-03
23	1,066131E-03	1,681529E-03	2,229959E-03	2,746019E-03
24	1,066131E-03	1,681529E-03	2,229959E-03	2,746019E-03
25	1,066131E-03	1,681529E-03	2,229959E-03	2,746019E-03

Table 2 pertains to the setup in Section II.1 and presents the results of the valuation of an at-the-money call within the Longstaff and Schwartz (1995) (L&S) framework for different quadrature rule orders *n*. The true price is given by the L&S analytic formula. The option's parameters are $X = \ln(0.02)$; $K = 0.02$; $r = 0.05$ and $V = 0.04$. To match the L&S framework, the model's parameters are $\mu = 0.02$; $\alpha = 0.015$; $\gamma = 0$; $\rho = 0$; $\sigma = 0$; $\lambda = 0$; $\kappa = 0$ and $\theta = 0$.

Table 3: Convergence to the Zhu square-root call price

Maturity	3 months	6 months	9 months	1 year
Zhu price	4,962005	7,620725	9,824956	11,766004
n				
7	4,947824	7,620594	9,825099	11,765874
8	4,961141	7,620671	9,824932	11,766032
9	4,962557	7,620737	9,824959	11,765998
10	4,962208	7,620725	9,824955	11,766006
11	4,962029	7,620725	9,824956	11,766004
12	4,962002	7,620725	9,824956	11,766004
13	4,962004	7,620725	9,824956	11,766004
14	4,962005	7,620725	9,824956	11,766004
15	4,962005	7,620725	9,824956	11,766004
16	4,962005	7,620725	9,824956	11,766004
17	4,962005	7,620725	9,824956	11,766004
18	4,962005	7,620725	9,824956	11,766004
19	4,962005	7,620725	9,824956	11,766004
20	4,962005	7,620725	9,824956	11,766004
21	4,962005	7,620725	9,824956	11,766004
22	4,962005	7,620725	9,824956	11,766004
23	4,962005	7,620725	9,824956	11,766004
24	4,962005	7,620725	9,824956	11,766004
25	4,962005	7,620725	9,824956	11,766004

Table 3 pertains to the setup in Section II.1 and presents the results of the valuation of an at-the-money call within the Zhu (2000) square-root framework for different quadrature rule orders n. The true price is given by the Zhu square-root semi-analytic formula. The option's parameters are $X = \ln(100)$; $K = 100$; $r = 0.05$ and $V = 0.04$. To match the Zhu square-root framework, the model's parameters are $\mu = r$; $\alpha = 0$; $\gamma = 0.5$; $\rho = -0.5$; $\sigma = 0.1$; $\lambda = 4$; $\kappa = 4$ and $\theta = 0.06$.

Table 4: Convergence to the Zhu Ornstein-Uhlenbeck call price

Maturity	3 months	6 months	9 months	1 year
Zhu price	3,692764	4,977335	6,056673	7,089761
n				
7	3,524815	4,940583	6,081476	7,143954
8	3,619622	4,985383	6,089360	7,124454
9	3,668585	4,994529	6,077641	7,103581
10	3,690008	4,990486	6,065813	7,092411
11	3,696937	4,984396	6,059099	7,088899
12	3,697505	4,980122	6,056573	7,088730
13	3,696076	4,977996	6,056122	7,089273
14	3,694563	4,977249	6,056309	7,089642
15	3,693544	4,977128	6,056528	7,089772
16	3,693010	4,977196	6,056643	7,089787
17	3,692792	4,977273	6,056680	7,089775
18	3,692731	4,977317	6,056683	7,089765
19	3,692730	4,977335	6,056679	7,089761
20	3,692744	4,977339	6,056676	7,089761
21	3,692755	4,977338	6,056674	7,089761
22	3,692761	4,977337	6,056674	7,089761
23	3,692763	4,977336	6,056673	7,089761
24	3,692764	4,977336	6,056673	7,089761
25	3,692764	4,977335	6,056673	7,089761

Table 4 pertains to the setup in Section II.2 and presents the results of the valuation of an at-the-money call within the Zhu Ornstein-Uhlenbeck (2000) framework for different quadrature rule orders n. The true price is given by the Zhu Ornstein-Uhlenbeck semi-analytic formula. The option's parameters are $X = \ln(100)$; $K = 100$; $r = 0.05$ and $\sigma = 0.2$. To match the Zhu Ornstein-Uhlenbeck framework, the model's parameters are $\mu = r$; $\alpha = 0$; $\gamma = 0.5$; $\rho = -0.5$; $\beta = 0.1$; $\lambda = 4$; $\kappa = 4$ and $\theta = 0.06$.

Table 5: Convergence of a call price within our
square-root mean reverting model

Maturity	3 months	6 months	9 months	1 year
n				
7	1,174022E-03	1,924637E-03	2,617826E-03	3,294297E-03
8	1,176620E-03	1,921935E-03	2,618714E-03	3,294585E-03
9	1,175539E-03	1,921684E-03	2,619036E-03	3,294459E-03
10	1,174195E-03	1,921888E-03	2,619034E-03	3,294433E-03
11	1,173450E-03	1,921994E-03	2,619009E-03	3,294439E-03
12	1,173187E-03	1,922013E-03	2,619004E-03	3,294441E-03
13	1,173142E-03	1,922009E-03	2,619005E-03	3,294441E-03
14	1,173156E-03	1,922006E-03	2,619005E-03	3,294441E-03
15	1,173171E-03	1,922005E-03	2,619006E-03	3,294441E-03
16	1,173177E-03	1,922005E-03	2,619005E-03	3,294441E-03
17	1,173179E-03	1,922005E-03	2,619005E-03	3,294441E-03
18	1,173179E-03	1,922005E-03	2,619005E-03	3,294441E-03
19	1,173179E-03	1,922005E-03	2,619005E-03	3,294441E-03
20	1,173179E-03	1,922005E-03	2,619005E-03	3,294441E-03
21	1,173179E-03	1,922005E-03	2,619005E-03	3,294441E-03
22	1,173179E-03	1,922005E-03	2,619005E-03	3,294441E-03
23	1,173179E-03	1,922005E-03	2,619005E-03	3,294441E-03
24	1,173179E-03	1,922005E-03	2,619005E-03	3,294441E-03
25	1,173179E-03	1,922005E-03	2,619005E-03	3,294441E-03

Table 5 pertains to the setup in Section II.1 and presents the results of the valuation of an at-the-money call within our square-root mean reverting framework for different quadrature rule orders n. The option's parameters are $X = \ln(0.02)$; $K = 0.02$; $r = 0.05$ and $V = 0.04$. The model's parameters are $\mu = 0.03$; $\alpha = 0.02$; $\gamma = 0$; $\rho = -0.5$; $\sigma = 0.2$; $\lambda = 1$; $\kappa = 1$ and $\theta = 0.05$.

**Table 6: Convergence of a call price within our
Ornstein-Uhlenbeck mean reverting model**

Maturity	3 months	6 months	9 months	1 year
n				
7	1,124247E-03	1,824871E-03	2,439173E-03	3,024480E-03
8	1,132712E-03	1,817122E-03	2,429618E-03	3,021436E-03
9	1,134126E-03	1,810972E-03	2,426195E-03	3,022995E-03
10	1,132717E-03	1,807384E-03	2,425925E-03	3,024767E-03
11	1,130681E-03	1,805781E-03	2,426633E-03	3,025593E-03
12	1,128934E-03	1,805327E-03	2,427299E-03	3,025725E-03
13	1,127718E-03	1,805388E-03	2,427659E-03	3,025613E-03
14	1,126987E-03	1,805596E-03	2,427775E-03	3,025504E-03
15	1,126604E-03	1,805783E-03	2,427770E-03	3,025458E-03
16	1,126439E-03	1,805904E-03	2,427732E-03	3,025453E-03
17	1,126392E-03	1,805963E-03	2,427702E-03	3,025461E-03
18	1,126401E-03	1,805983E-03	2,427688E-03	3,025468E-03
19	1,126429E-03	1,805983E-03	2,427684E-03	3,025471E-03
20	1,126458E-03	1,805976E-03	2,427685E-03	3,025471E-03
21	1,126481E-03	1,805970E-03	2,427687E-03	3,025470E-03
22	1,126496E-03	1,805965E-03	2,427688E-03	3,025470E-03
23	1,126506E-03	1,805963E-03	2,427689E-03	3,025470E-03
24	1,126511E-03	1,805962E-03	2,427689E-03	3,025470E-03
25	1,126513E-03	1,805962E-03	2,427689E-03	3,025470E-03

Table 6 pertains to the setup in Section II.2 and presents the results of the valuation of an at-the-money call within our Ornstein-Uhlenbeck mean reverting framework for different quadrature rule orders n. The option's parameters are $X = \ln(0.02)$; $K = 0.02$; $r = 0.05$ and $\sigma = 0.2$. The model's parameters are $\mu = 0.03$; $\alpha = 0.02$; $\gamma = 0$; $\rho = -0.5$; $\beta = 0.2$; $\lambda = 1$; $\kappa = 1$ and $\theta = 0.05$.

Table 7: Impact of mean reversion on the call price within our square-root mean reverting model

		Maturity			
		3 months	6 months	9 months	1 year
	α				
OTM	0.01	19%	22%	25%	26%
	0.02	34%	38%	42%	44%
	0.03	45%	50%	54%	56%
	α				
ATM	0.01	11%	15%	18%	20%
	0.02	21%	28%	32%	35%
	0.03	29%	37%	42%	46%
	α				
ITM	0.01	4%	7%	10%	12%
	0.02	8%	14%	18%	22%
	0.03	12%	20%	25%	30%

Table 7 pertains to the setup in Section II.1 and presents the impact of the mean reversion coefficient on the call price within our square-root mean reverting framework. The relative difference is computed with respect to the "no mean reversion call price" i.e. $\alpha = 0$. The underlying asset log-values are $X = \ln(0.02)$; $X = \ln(0.025)$ and $X = \ln(0.018)$. The option's parameters are $K = 0.02$; $r = 0.05$ and $V = 0.04$. The model's parameters are $\mu = 0.03$; $\gamma = 0$; $\rho = -0.5$; $\sigma = 0.2$; $\lambda = 1$; $\kappa = 1$ and $\theta = 0.05$.

Table 8: Impact of mean reversion on the call price within our Ornstein-Uhlenbeck mean reverting model

		Maturity			
		3 months	6 months	9 months	1 year
	α				
OTM	0.01	22%	27%	31%	34%
	0.02	38%	46%	50%	54%
	0.03	50%	58%	63%	66%
	α				
ATM	0.01	12%	18%	21%	24%
	0.02	23%	31%	36%	40%
	0.03	31%	41%	47%	52%
	α				
ITM	0.01	4%	7%	10%	13%
	0.02	8%	14%	19%	23%
	0.03	12%	20%	26%	31%

Table 8 pertains to the setup in Section II.2 and presents the impact of the mean reversion coefficient on the call price within our Ornstein-Uhlenbeck mean reverting framework. The relative difference is computed with respect to the "no mean reversion call price" i.e. $\alpha = 0$. The underlying asset log-values are $X = \ln(0.02)$; $X = \ln(0.025)$ and $X = \ln(0.018)$. The option's parameters are $K = 0.02$; $r = 0.05$ and $V = 0.04$. The model's parameters are $\mu = 0.03$; $\gamma = 0$; $\rho = -0.5$; $\beta = 0.2$; $\lambda = 1$; $\kappa = 1$ and $\theta = 0.05$.

Figures

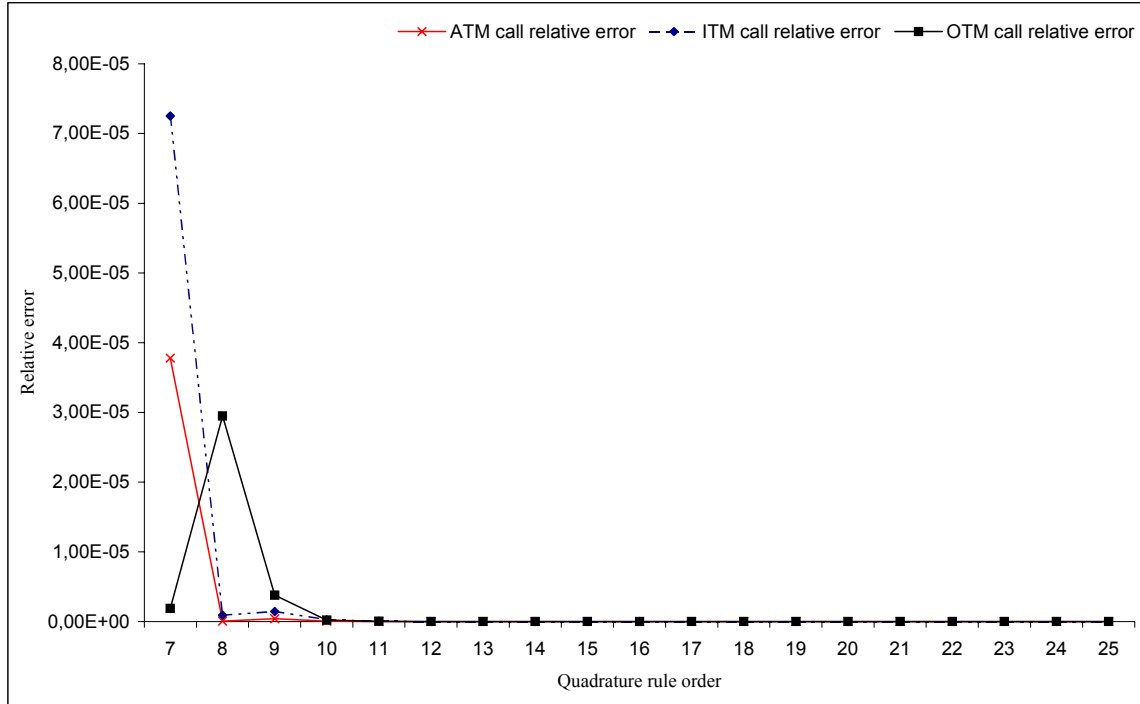## Figure 1: Relative pricing error with respect to the Black-Scholes price

Figure 1 shows the relative pricing error of a call within the Black and Scholes (1973) (B&S) framework. The true price is given by the B&S analytic formula. The relative pricing error is computed as $\dfrac{\left|\text{Numerical price} - \text{B \& S price}\right|}{\text{B \& S price}}$. The underlying asset log-values are $X = \ln(100)$; $X = \ln(110)$ and $X = \ln(90)$. The option's parameters are $K = 100$; $T = 0.5$; $r = 0.05$ and $V = 0.04$. To match the B&S framework, the model's parameters are $\mu = r$; $\alpha = 0$; $\gamma = 0.5$; $\rho = 0$; $\sigma = 0$; $\lambda = 0$; $\kappa = 0$ and $\theta = 0$.

Figure 2: Relative pricing error with respect to the Longstaff-Schwartz price
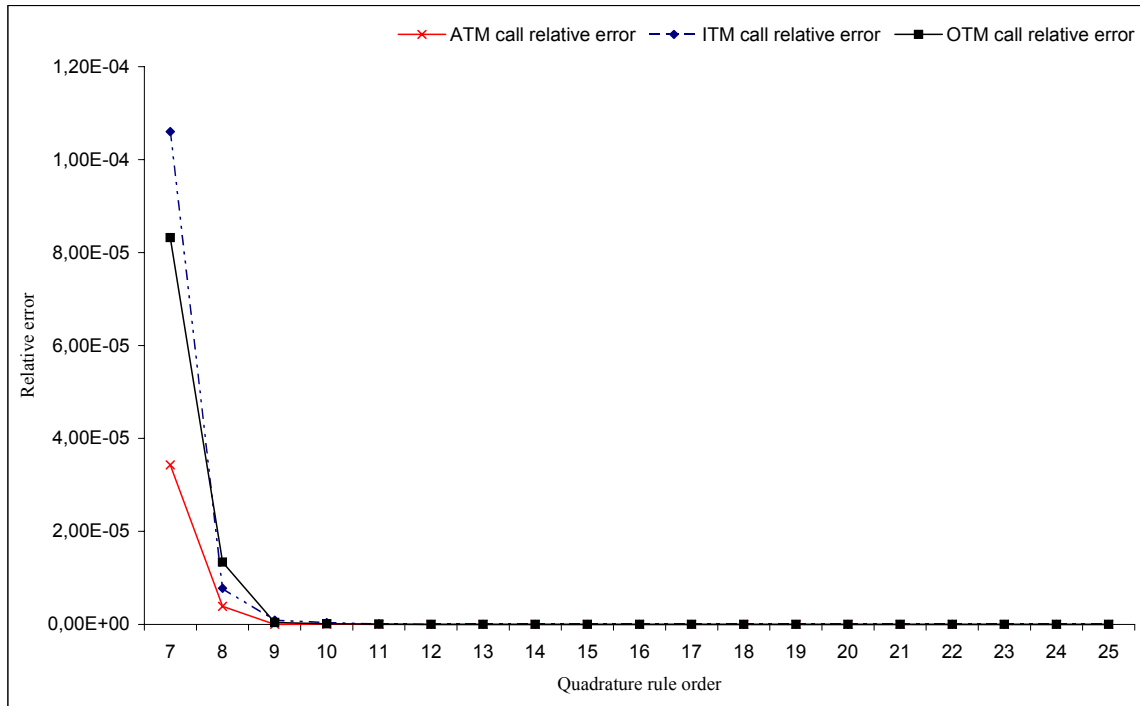

Figure 2 shows the relative pricing error of a call within the Longstaff and Schwartz (1995) (L&S) framework. The true price is given by the L&S analytic formula. The relative pricing

error is computed as $\dfrac{\left|\text{Numerical price} - \text{L \& S price}\right|}{\text{L \& S price}}$. The underlying asset log-values are

$X = \ln(0.02)$; $X = \ln(0.022)$ and $X = \ln(0.018)$. The option's parameters are $K = 0.02$; $T = 0.5$; $r = 0.05$ and $V = 0.04$. To match the L&S framework, the model's parameters are $\mu = 0.02$; $\alpha = 0.015$; $\gamma = 0$; $\rho = 0$; $\sigma = 0$; $\lambda = 0$; $\kappa = 0$ and $\theta = 0$.

Figure 3: Relative pricing error with respect to the Zhu square-root price
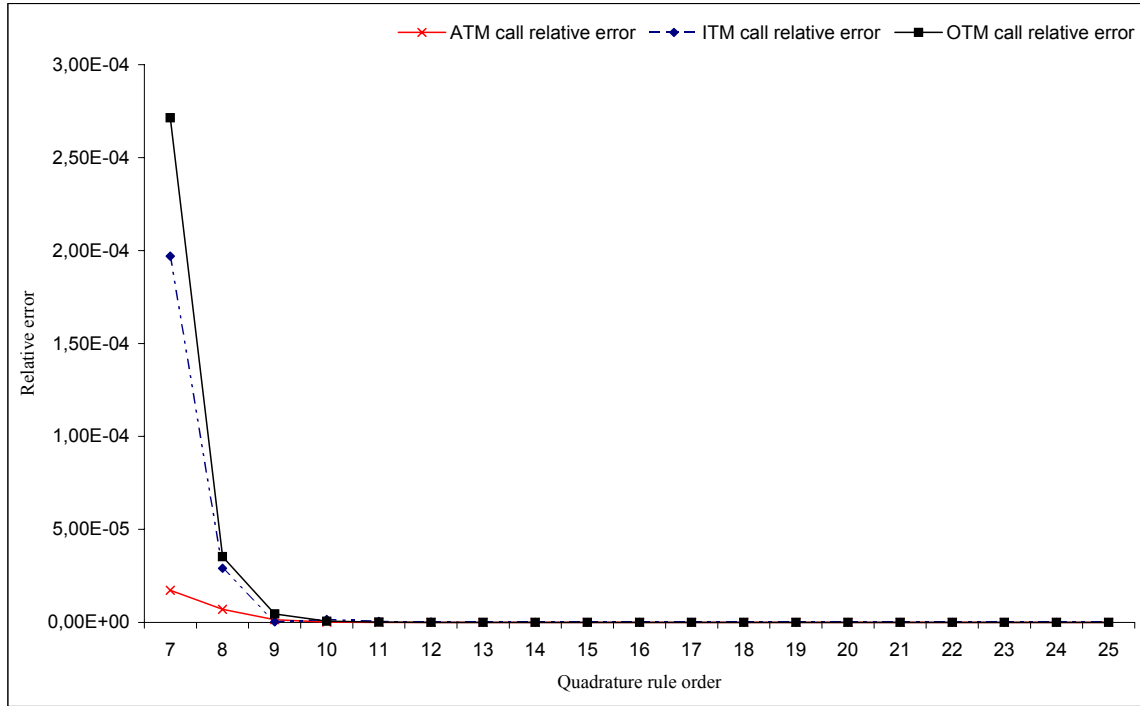

Figure 3 shows the relative pricing error of a call within the Zhu (2000) square-root framework. The true price is given by the Zhu square-root semi-analytic formula. The relative pricing error is computed as $\dfrac{\left|\text{Numerical price} - \text{Zhu price}\right|}{\text{Zhu price}}$. The underlying asset log-values are $X = \ln(100)$; $X = \ln(120)$ and $X = \ln(80)$. The option's parameters are $K = 100$; $T = 0.5$; $r = 0.05$ and $V = 0.04$. To match the Zhu square-root framework, the model's parameters are $\mu = 0.05$; $\alpha = 0$; $\gamma = 0.5$; $\rho = -0.5$; $\sigma = 0.1$; $\lambda = 4$; $\kappa = 4$ and $\theta = 0.06$.

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Figure 4: Relative pricing error with respect to the Zhu Ornstein-Uhlenbeck price
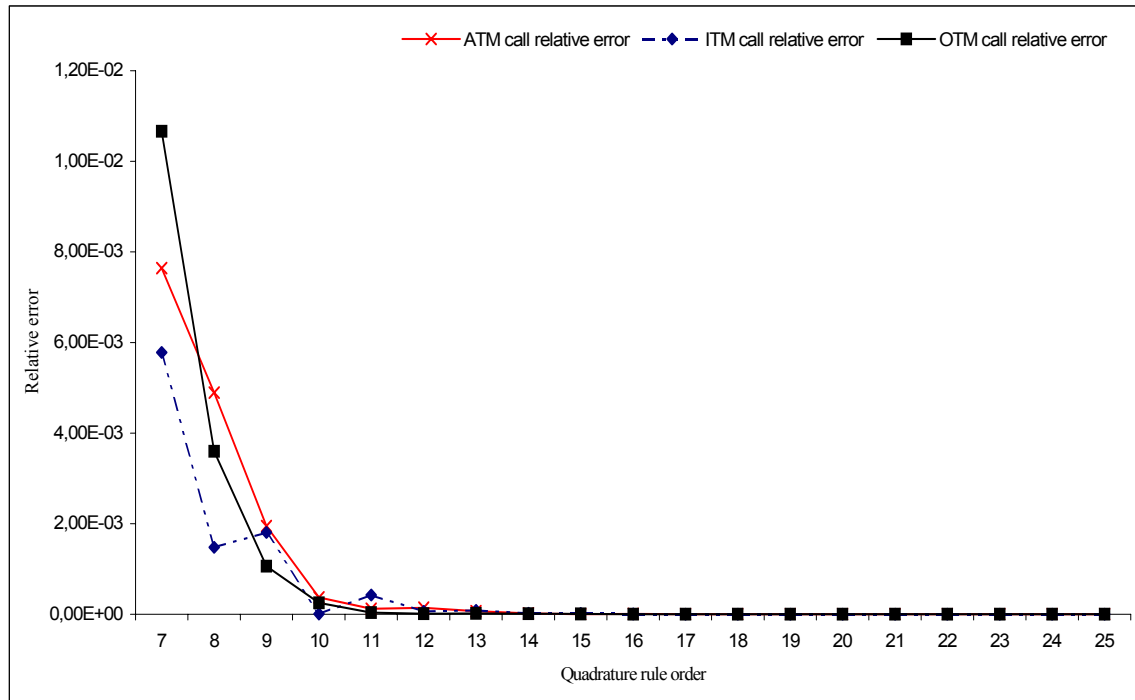

Figure 4 shows the relative pricing error of a call within the Zhu (2000) Ornstein-Uhlenbeck framework. The true price is given by the Zhu Ornstein-Uhlenbeck semi-analytic formula. The relative pricing error is computed as $\dfrac{|\text{Numerical price} - \text{Zhu price}|}{\text{Zhu price}}$. The underlying asset log-values are $X = \ln(100)$; $X = \ln(120)$ and $X = \ln(80)$. The option's parameters are $K = 100$; $T = 1$; $r = 0.05$ and $\sigma = 0.2$. To match the Zhu Ornstein-Uhlenbeck framework, the model's parameters are $\mu = 0.05$; $\alpha = 0$; $\gamma = 0.5$; $\rho = -0.5$; $\beta = 0.1$; $\lambda = 4$; $\kappa = 4$ and $\theta = 0.06$. To keep the same scale, the out-of-the-money pricing relative error is divided by 50.

Figure 5: Relative pricing error within our square-root mean reverting model
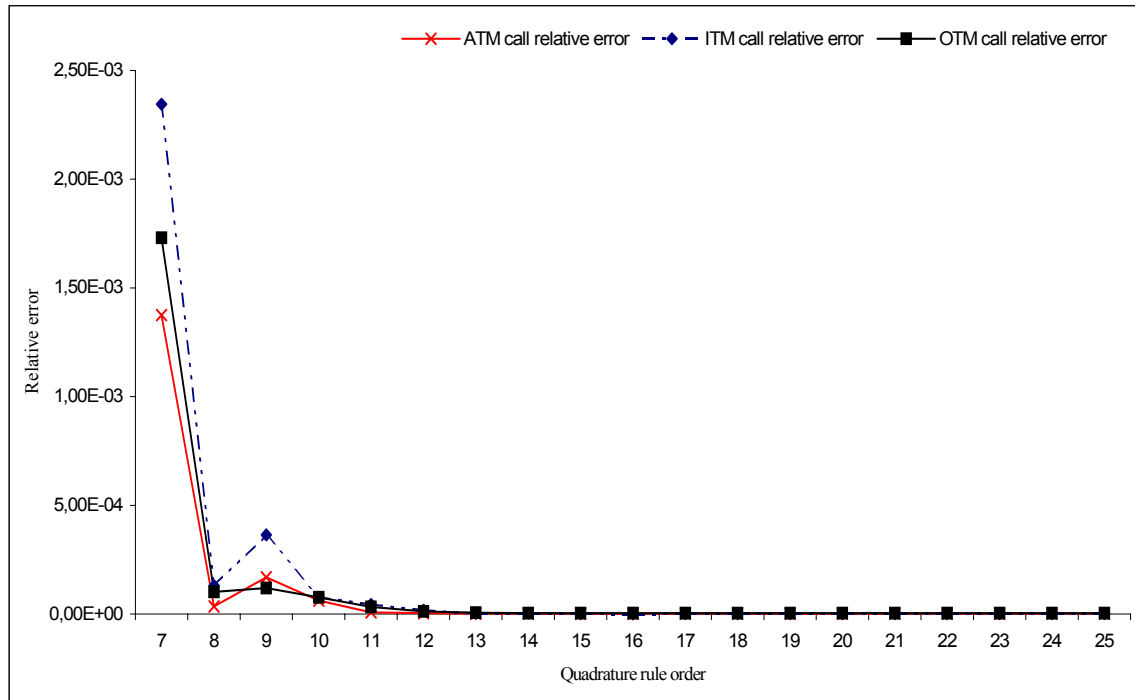

Figure 5 shows the relative pricing error of a call within our square-root mean reverting framework. The true price is the asymptotic price. The relative pricing error is computed as $\dfrac{\left| \text{Numerical price} - \text{Asymptotic price} \right|}{\text{Asymptotic price}}$. The underlying asset log-values are $X = \ln(0.02)$; $X = \ln(0.025)$ and $X = \ln(0.018)$. The option's parameters are $K = 0.02$; $T = 0.5$; $r = 0.05$ and $V = 0.04$. The model's parameters are $\mu = 0.03$; $\alpha = 0.02$; $\gamma = 0$; $\rho = -0.5$; $\sigma = 0.2$; $\lambda = 1$; $\kappa = 1$ and $\theta = 0.05$.

Figure 6: Relative pricing error within our Ornstein-Uhlenbeck mean reverting model
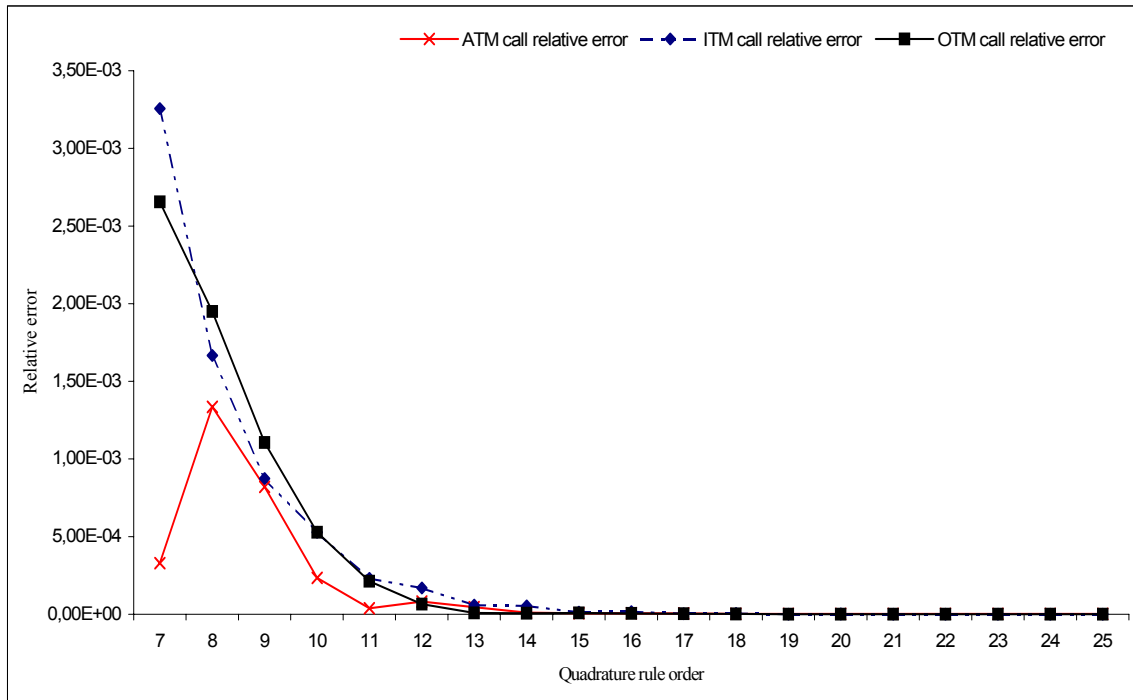

Figure 6 shows the relative pricing error of a call within our Ornstein-Uhlenbeck mean reverting framework. The true price is the asymptotic price. The relative pricing error is computed as $\dfrac{|\text{Numerical price} - \text{Asymptotic price}|}{\text{Asymptotic price}}$. The underlying asset log-values are $X = \ln(0.02)$; $X = \ln(0.025)$ and $X = \ln(0.018)$. The option's parameters are $K = 0.02$; $T = 1$; $r = 0.05$ and $\sigma = 0.2$. The model's parameters are $\mu = 0.03$; $\alpha = 0,02$; $\gamma = 0$; $\rho = -0.5$; $\beta = 0.2$; $\lambda = 1$; $\kappa = 1$ and $\theta = 0.05$.

Appendix A.1: Derivation of the square-root mean-reverting characteristic function

Let W_1 and W_2 be two correlated Brownian motions under Q with $d\langle W_1, W_2 \rangle_t = \rho dt$.

The model is given under the risk-neutral measure Q by:

$$dX_t = \left(\mu - \alpha X_t - \gamma V_t \right) dt + \sqrt{V_t} \, dW_1(t)$$

$$dV_t = \left(\kappa \theta - \lambda V_t \right) dt + \sigma \sqrt{V_t} \, dW_2(t)$$

If we define $Y_t = e^{\alpha t} X_t$, by Ito's lemma we have:

$$dY_t = e^{\alpha t} \left(\mu - \gamma V_t \right) dt + e^{\alpha t} \sqrt{V_t} \, dW_1(t)$$

Solving this SDE gives:

$$Y_T = Y_t + \int_t^T e^{\alpha s} \left(\mu - \gamma V_s \right) ds + \int_t^T e^{\alpha s} \sqrt{V_s} \, dW_1(s)$$

$$Y_T = Y_t + \frac{\mu}{\alpha} \left(e^{\alpha T} - e^{\alpha t} \right) - \gamma \int_t^T e^{\alpha s} V_s \, ds + \int_t^T e^{\alpha s} \sqrt{V_s} \, dW_1(s)$$

The process X is then expressed as:

$$X_T = e^{-\alpha(T-t)} X_t + \frac{\mu}{\alpha} \left(1 - e^{-\alpha(T-t)} \right) - \gamma \int_t^T e^{-\alpha(T-s)} V_s \, ds + \int_t^T e^{-\alpha(T-s)} \sqrt{V_s} \, dW_1(s)$$

So we have that:

$$\exp(\psi X_T) = \exp\left(\psi e^{-\alpha(T-t)} X_t + \frac{\mu}{\alpha} \psi \left(1 - e^{-\alpha(T-t)} \right) \right) \times \exp\left(-\gamma \psi \int_t^T e^{-\alpha(T-s)} V_s \, ds \right)$$

$$\times \exp\left(\psi \int_t^T e^{-\alpha(T-s)} \sqrt{V_s} \, dW_1(s) \right)$$

Since W_1 and W_2 are correlated, we can write $dW_1(s) = \rho dW_2(s) + \sqrt{1 - \rho^2} \, dW(s)$, where W is a Brownian motion uncorrelated with W_2. Let E_t^Q denote mathematical expectation taken under the probability measure Q conditioned on the information up to time t. We then obtain:

$$E_t^Q\left(e^{\psi X_T}\right) = \exp\left(\psi e^{-\alpha(T-t)} X_t + \frac{\mu}{\alpha}\psi\left(1 - e^{-\alpha(T-t)}\right)\right)$$

$$\times E_t^Q \left(\begin{array}{l} \exp\left(-\gamma\psi\int_t^T e^{-\alpha(T-s)}V_s ds\right) \times \exp\left(\rho\psi\int_t^T e^{-\alpha(T-s)}\sqrt{V_s}dW_2(s)\right) \\ \times \exp\left(\psi\sqrt{1-\rho^2}\int_t^T e^{-\alpha(T-s)}\sqrt{V_s}dW(s)\right) \end{array}\right)$$

$$= \exp\left(\psi e^{-\alpha(T-t)} X_t + \frac{\mu}{\alpha}\psi\left(1 - e^{-\alpha(T-t)}\right)\right)$$

$$\times E_t^Q \left(\begin{array}{l} \exp\left(-\gamma\psi\int_t^T e^{-\alpha(T-s)}V_s ds\right) \times \exp\left(\rho\psi\int_t^T e^{-\alpha(T-s)}\sqrt{V_s}dW_2(s)\right) \\ \times E_t^Q\left[\exp\left(\psi\sqrt{1-\rho^2}\int_t^T e^{-\alpha(T-s)}\sqrt{V_s}dW(s)\right)\middle| \mathcal{F}(W_2(s):t\leq s\leq T)\right] \end{array}\right)$$

where $\mathcal{F}(W_2(s):t\leq s\leq T)$ represents the filtration generated by $(W_2(s):t\leq s\leq T)$. Since W and W_2 are independent, the equation becomes:

$$E_t^Q\left(e^{\psi X_T}\right) = \exp\left(\psi e^{-\alpha(T-t)} X_t + \frac{\mu}{\alpha}\psi\left(1 - e^{-\alpha(T-t)}\right)\right)$$

$$\times E_t^Q \left(\begin{array}{l} \exp\left(-\gamma\psi\int_t^T e^{-\alpha(T-s)}V_s ds + \frac{1}{2}\psi^2\left(1-\rho^2\right)\int_t^T e^{-2\alpha(T-s)}V_s ds\right) \\ \times \exp\left(\rho\psi\int_t^T e^{-\alpha(T-s)}\sqrt{V_s}dW_2(s)\right) \end{array}\right)$$

At this stage, we did not need the particular square-root specification of the volatility diffusion. These equations will be also valid for the Ornstein-Uhlenbeck volatility diffusion. For the square-root model, by Ito's lemma we can write for the squared volatility:

$$d\left(e^{-\alpha(T-t)}V_t\right) = e^{-\alpha(T-t)}\left(\kappa\theta + (\alpha - \lambda)V_t\right)dt + e^{-\alpha(T-t)}\sigma\sqrt{V_t}dW_2(t)$$

Integrating this SDE and re-arranging it leads to:

$$\sigma\int_t^T e^{-\alpha(T-s)}\sqrt{V_s}dW_2(s) = V_T - e^{-\alpha(T-t)}V_t - \int_t^T e^{-\alpha(T-s)}\left(\kappa\theta + (\alpha - \lambda)V_s\right)ds$$

We then obtain:

$$E_t^Q\left(e^{\psi X_T}\right)=\exp\left(\psi e^{-\alpha(T-t)}X_t+\frac{\mu}{\alpha}\psi\left(1-e^{-\alpha(T-t)}\right)\right)$$

$$\times E_t^Q\left(\begin{array}{l}\exp\left(-\gamma\psi\int_t^T e^{-\alpha(T-s)}V_s\,ds+\frac{1}{2}\psi^2\left(1-\rho^2\right)\int_t^T e^{-2\alpha(T-s)}V_s\,ds\right)\\[4pt]\times\exp\left(-\frac{\rho}{\sigma}\psi e^{-\alpha(T-t)}V_t-\frac{\rho}{\sigma}\frac{\kappa\theta}{\alpha}\psi\left(1-e^{-\alpha(T-t)}\right)\right)\\[4pt]\times\exp\left(\frac{\rho}{\sigma}\psi V_T-\frac{\rho}{\sigma}\psi(\alpha-\lambda)\int_t^T e^{-\alpha(T-t)}V_s\,ds\right)\end{array}\right)$$

$$=\exp\left(\psi e^{-\alpha(T-t)}X_t+\frac{\mu}{\alpha}\psi\left(1-e^{-\alpha(T-t)}\right)-\frac{\rho}{\sigma}\psi e^{-\alpha(T-t)}V_t-\frac{\rho}{\sigma}\frac{\kappa\theta}{\alpha}\psi\left(1-e^{-\alpha(T-t)}\right)\right)$$

$$\times E_t^Q\left(\exp\left(\begin{array}{l}\frac{\rho}{\sigma}\psi V_T\\[4pt]+\int_t^T\left[\frac{1}{2}\psi^2\left(1-\rho^2\right)e^{-2\alpha(T-s)}-\left(\frac{\rho}{\sigma}(\alpha-\lambda)+\gamma\right)\psi e^{-\alpha(T-s)}\right]V_s\,ds\end{array}\right)\right)$$

We can rewrite this equation as:

$$E_t^Q\left(e^{\psi X_T}\right)=\exp\left(\psi e^{-\alpha(T-t)}X_t+\frac{\mu}{\alpha}\psi\left(1-e^{-\alpha(T-t)}\right)-\frac{\rho}{\sigma}\psi e^{-\alpha(T-t)}V_t-\frac{\rho}{\sigma}\frac{\kappa\theta}{\alpha}\psi\left(1-e^{-\alpha(T-t)}\right)\right)$$

$$\times E_t^Q\left(\exp\left(\varepsilon_2 V_T-\int_t^T\varepsilon_1(T-s)V_s\,ds\right)\right)$$

where

$$\left\{\begin{array}{l}\varepsilon_1(\tau)=\left(\frac{\rho}{\sigma}(\alpha-\lambda)+\gamma\right)\psi\exp(-\alpha\tau)-\frac{1}{2}\psi^2\left(1-\rho^2\right)\exp(-2\alpha\tau)\\[8pt]\varepsilon_2=\frac{\rho}{\sigma}\psi\end{array}\right.$$

Define the function $F(t,V)$ by:

$$F(t,V)=E_{t,V}^Q\left[\exp\{\varepsilon_2 V_T\}\exp\left\{-\int_t^T\varepsilon_1(T-s)V_s\,ds\right\}\right]$$

then by Feynman-Kac theorem (see Appendix D), we have that $F(t,V)$ must satisfy the following PDE:

$$\begin{cases} \dfrac{\partial F}{\partial t} + \dfrac{1}{2}\sigma^2 V \dfrac{\partial^2 F}{\partial V^2} + (\kappa\theta - \lambda V)\dfrac{\partial F}{\partial V} - \varepsilon_1(T-t)V\,F = 0 \\[4mm] F(T,V) = \exp(\varepsilon_2 V) \end{cases}$$

Replacing the time variable t by $\tau = T - t$, we can rewrite this PDE as (without ambiguity, we keep the same function's name F):

$$\begin{cases} \dfrac{\partial F}{\partial \tau} = \dfrac{1}{2}\sigma^2 V \dfrac{\partial^2 F}{\partial V^2} + (\kappa\theta - \lambda V)\dfrac{\partial F}{\partial V} - \varepsilon_1(\tau)V\,F \\[4mm] F(0,V) = \exp(\varepsilon_2 V) \end{cases}$$

If we assume that $F(\tau,V)$ is log-linear and given by:

$$F(\tau,V) = \exp[D(\tau)V + C(\tau)]$$

where

$$D(0) = \varepsilon_2 \quad ; \quad C(0) = 0$$

we have:

$$\begin{cases} \dfrac{\partial F}{\partial \tau} = [D'(\tau)V + C'(\tau)] \times F(\tau,V) \\[4mm] \dfrac{\partial F}{\partial V} = D(\tau) \times F(\tau,V) \\[4mm] \dfrac{\partial^2 F}{\partial V^2} = D^2(\tau) \times F(\tau,V) \end{cases}$$

The PDE for $F(\tau,V)$ becomes:

$$D'(\tau)V + C'(\tau) = \dfrac{1}{2}\sigma^2 D^2(\tau)V + (\kappa\theta - \lambda V)D(\tau) - V\varepsilon_1(\tau)$$

After re-arranging it as a polynomial of V, we deduce the ODEs satisfied by $D(\tau)$:

$$\begin{cases} D'(\tau) - \dfrac{1}{2}\sigma^2 D^2(\tau) + \lambda D(\tau) + \varepsilon_1(\tau) = 0 \\[4mm] D(0) = \varepsilon_2 = \dfrac{\rho}{\sigma}\psi \end{cases}$$

and by $C(\tau)$:

$$\begin{cases} C'(\tau) - \kappa\theta\, D(\tau) = 0 \\[2mm] C(0) = 0 \end{cases}$$

The actualized characteristic function is then given by:

$$f(\psi) \equiv E_t^Q\left[\exp\left(-\int_t^T r(s)ds \right) \exp(\psi X_T) \right]$$

$$= e^{-r(T-t)} \times \exp\left(\psi e^{-\alpha(T-t)} X_t + \frac{\mu}{\alpha}\psi\left(1 - e^{-\alpha(T-t)}\right) - \frac{\rho}{\sigma}\psi e^{-\alpha(T-t)} V_t - \frac{\rho}{\sigma}\frac{\kappa\theta}{\alpha}\psi\left(1 - e^{-\alpha(T-t)}\right) \right)$$

$$\times \exp\left[D(T-t;\psi)V_t + C(T-t;\psi) \right]$$

Appendix A.2: Exact resolution of the ODEs satisfied by *D* and *C* in the square-root framework

The ODEs satisfied by the functions *D* and *C* are:

$$\begin{cases} D'(\tau) - \dfrac{1}{2}\sigma^2 D^2(\tau) + \lambda\, D(\tau) + \varepsilon_1(\tau) = 0 \\[3mm] D(0) = \dfrac{\rho}{\sigma}\psi \end{cases}$$

and

$$\begin{cases} C'(\tau) - \kappa\theta\, D(\tau) = 0 \\[2mm] C(0) = 0 \end{cases}$$

Making the traditional (for Riccati-type ODEs) following transformation:

$$U(\tau) = \exp\left(-\frac{\sigma^2}{2}\int D(s)ds \right)$$

leads to the following linear homogeneous second-order ODE:

$$U''(\tau) + \lambda U'(\tau) - \frac{1}{2}\sigma^2 U(\tau)\varepsilon_1(\tau) = 0$$

Under this transformation we recover the original functions *D* and *C* simply by:

$$\begin{cases} D(\tau) = -\dfrac{2}{\sigma^2}\dfrac{U'(\tau)}{U(\tau)} \\[3mm] C(\tau) = -\dfrac{2\kappa\theta}{\sigma^2}\ln\big(U(\tau)\big) \end{cases}$$

A further substitution $V(z) \equiv V\big(\exp(-\alpha\tau)\big) \equiv U(\tau)$ reduces the ODE to:

$$\alpha^2 z^2 V''(z) + \alpha(\alpha - \lambda)z\,V'(\tau) - \frac{1}{2}\sigma^2 V(z)\,e_1(z) = 0$$

where $e_1(z) \equiv \varepsilon_1(\tau)$. We then only need to solve for the function $V(z)$. Softwares like $Maple^{®}$ give the solution to this ODE in terms of special functions known as the Whittaker functions. These functions are linked to the well-known hypergeometric functions (see Abramowitz and Stegun, 1968). The solution U is given by:

$$\begin{aligned} U(\tau) &\equiv V(e^{-\alpha\tau}) \\ &= A\exp\big(-d\alpha\tau\big)M\big(a,b,c\psi e^{-\alpha\tau}\big) + B\exp\big(-d\alpha\tau\big)W\big(a,b,c\psi e^{-\alpha\tau}\big) \end{aligned}$$

where $M(.)$ and $W(.)$ are respectively the *WhittakerM* and the *WhittakerW* functions and:

$$\begin{cases} a = -\dfrac{\rho\alpha + \gamma\sigma - \rho\lambda}{2\alpha\sqrt{\rho^2 - 1}} \quad & ; \quad b = \dfrac{\lambda}{2\alpha} \\[3mm] c = \dfrac{\sigma\sqrt{\rho^2 - 1}}{\alpha} \quad & ; \quad d = \dfrac{\lambda - \alpha}{2\alpha} \end{cases}$$

Constants A and B are determined by writing down that $D(0) = \dfrac{\rho}{\sigma}\psi$ and $C(0) = 0$.

Appendix B.1: Derivation of the Ornstein-Uhlenbeck mean-reverting characteristic function

Let W_1 and W_2 be two correlated Brownian motions under Q with $d\langle W_1, W_2\rangle_t = \rho dt$. The model is given under the risk-neutral measure Q by:

$$dX_t = \big(\mu - \alpha X_t - \gamma\sigma_t^2\big)dt + \sigma_t dW_1(t)$$

$$d\sigma_t = \big(\kappa\theta - \lambda\sigma_t\big)dt + \beta\, dW_2(t)$$

From Appendix A.1, we can write the characteristic function as:

$$E_t^Q\left(e^{\psi X_T}\right) = \exp\left(\psi e^{-\alpha(T-t)} X_t + \frac{\mu}{\alpha}\psi\left(1 - e^{-\alpha(T-t)}\right)\right)$$

$$\times E_t^Q\left(\begin{array}{l}\exp\left(-\gamma\psi\int_t^T e^{-\alpha(T-s)}\sigma_s^2 ds + \frac{1}{2}\psi^2\left(1-\rho^2\right)\int_t^T e^{-2\alpha(T-s)}\sigma_s^2 ds\right)\\ \times \exp\left(\rho\psi\int_t^T e^{-\alpha(T-s)}\sigma_s dW_2(s)\right)\end{array}\right)$$

For the Ornstein-Uhlenbeck model, we can solve for the volatility:

$$e^{-\alpha(T-s)}\sigma_s dW_2(s) = e^{-\alpha(T-s)}\frac{\sigma_s}{\beta}d\sigma_s - e^{-\alpha(T-s)}\frac{\kappa\theta}{\beta}\sigma_s ds + e^{-\alpha(T-s)}\frac{\lambda}{\beta}\sigma_s^2 ds$$

Integrating this SDE and re-arranging it leads to:

$$\int_t^T e^{-\alpha(T-s)}\sigma_s dW_2(s) = \frac{1}{\beta}\int_t^T e^{-\alpha(T-s)}\sigma_s d\sigma_s - \frac{\kappa\theta}{\beta}\int_t^T e^{-\alpha(T-s)}\sigma_s ds + \frac{\lambda}{\beta}\int_t^T e^{-\alpha(T-s)}\sigma_s^2 ds$$

By Ito's lemma, we also have:

$$d\left(e^{-\alpha(T-s)}\sigma_s^2\right) = \alpha e^{-\alpha(T-s)}\sigma_s^2 ds + 2e^{-\alpha(T-s)}\sigma_s d\sigma_s + \beta^2 e^{-\alpha(T-s)}ds$$

$$\frac{1}{\beta}\int_t^T e^{-\alpha(T-s)}\sigma_s d\sigma_s = \frac{1}{2\beta}\left[\sigma_T^2 - e^{-\alpha(T-t)}\sigma_t^2\right] - \frac{\alpha}{2\beta}\int_t^T e^{-\alpha(T-s)}\sigma_s^2 ds - \frac{\beta}{2\alpha}\left(1 - e^{-\alpha(T-t)}\right)$$

and

$$\rho\psi\int_t^T e^{-\alpha(T-s)}\sigma_s dW_2(s) = \rho\psi\left(\frac{1}{2\beta}\left[\sigma_T^2 - e^{-\alpha(T-t)}\sigma_t^2\right] - \frac{\alpha}{2\beta}\int_t^T e^{-\alpha(T-s)}\sigma_s^2 ds - \frac{\beta}{2\alpha}\left(1 - e^{-\alpha(T-t)}\right)\right)$$

$$- \frac{\rho\kappa\theta}{\beta}\psi\int_t^T e^{-\alpha(T-s)}\sigma_s ds + \rho\psi\frac{\lambda}{\beta}\int_t^T e^{-\alpha(T-s)}\sigma_s^2 ds$$

$$= \frac{\rho}{2\beta}\psi\left[\sigma_T^2 - e^{-\alpha(T-t)}\sigma_t^2\right] + \frac{\rho}{\beta}\psi\left(\lambda - \frac{\alpha}{2}\right)\int_t^T e^{-\alpha(T-s)}\sigma_s^2 ds$$

$$- \frac{\rho\beta}{2\alpha}\psi\left(1 - e^{-\alpha(T-t)}\right) - \frac{\rho\kappa\theta}{\beta}\psi\int_t^T e^{-\alpha(T-s)}\sigma_s ds$$

The characteristic function is then given by:

$$E_t^Q\left(e^{\psi X_T}\right)=\exp\left(\psi e^{-\alpha(T-t)}X_t+\frac{\mu}{\alpha}\psi\left(1-e^{-\alpha(T-t)}\right)-\frac{\rho}{2\beta}\psi e^{-\alpha(T-t)}\sigma_t^{\,2}-\frac{\rho\beta}{2\alpha}\psi\left(1-e^{-\alpha(T-t)}\right)\right)$$

$$\times E_t^Q\left(\exp\left(\begin{array}{c}\dfrac{\rho}{2\beta}\psi\sigma_T^{\,2}-\dfrac{\rho\kappa\theta}{\beta}\psi\displaystyle\int_t^T e^{-\alpha(T-s)}\sigma_s\,ds\\[2ex]+\displaystyle\int_t^T\left[\frac{1}{2}\psi^2\left(1-\rho^2\right)e^{-2\alpha(T-s)}-\left(\frac{\rho}{\beta}\left(\frac{\alpha}{2}-\lambda\right)+\gamma\right)\psi e^{-\alpha(T-s)}\right]\sigma_s^{\,2}\,ds\end{array}\right)\right)$$

We can rewrite this equation as:

$$E_t^Q\left(e^{\psi X_T}\right)=\exp\left(\psi e^{-\alpha(T-t)}X_t+\frac{\mu}{\alpha}\psi\left(1-e^{-\alpha(T-t)}\right)-\frac{\rho}{2\beta}\psi e^{-\alpha(T-t)}\sigma_t^{\,2}-\frac{\rho\beta}{2\alpha}\psi\left(1-e^{-\alpha(T-t)}\right)\right)$$

$$\times E_t^Q\left(\exp\left(\eta_3\sigma_T^{\,2}-\int_t^T\eta_2(T-s)\sigma_s\,ds-\int_t^T\eta_1(T-s)\sigma_s^{\,2}\,ds\right)\right)$$

where

$$\begin{cases}\eta_1(\tau)=\left(\dfrac{\alpha\rho}{2\beta}-\dfrac{\rho\lambda}{\beta}+\gamma\right)\psi\exp(-\alpha\tau)-\dfrac{1}{2}\psi^2\left(1-\rho^2\right)\exp(-2\alpha\tau)\\[2ex]\eta_2(\tau)=\dfrac{\rho\kappa\theta}{\beta}\psi\exp(-\alpha\tau)\\[2ex]\eta_3=\dfrac{\rho}{2\beta}\psi\end{cases}$$

Define the function $G(t,\sigma)$ as:

$$G(t,\sigma)=E_t^Q\left(\exp\left(\eta_3\sigma_T^{\,2}-\int_t^T\eta_2(T-s)\sigma_s\,ds-\int_t^T\eta_1(T-s)\sigma_s^{\,2}\,ds\right)\right)$$

By Feynman-Kac theorem, we have that $G(t,\sigma)$ must satisfy the following PDE:

$$\begin{cases}\dfrac{\partial G}{\partial t}+\dfrac{1}{2}\beta^2\dfrac{\partial^2 G}{\partial\sigma^2}+(\kappa\theta-\lambda\sigma)\dfrac{\partial G}{\partial\sigma}-\left(\eta_1(T-t)\sigma^2+\eta_2(T-t)\sigma\right)G=0\\[3ex]G(T,\sigma)=\exp\left(\eta_3\sigma^2\right)\end{cases}$$

Replacing the time variable t by $\tau=T-t$ as before, we can rewrite this PDE as (without ambiguity, we keep the same function's name G):

$$\begin{cases} \dfrac{\partial G}{\partial \tau} = \dfrac{1}{2}\beta^2 \dfrac{\partial^2 G}{\partial \sigma^2} + (\kappa\theta - \lambda\sigma)\dfrac{\partial G}{\partial \sigma} - \left(\eta_1(\tau)\sigma^2 + \eta_2(\tau)\sigma\right)G \\[4mm] G(0,\sigma) = \exp\!\left(\eta_3\sigma^2\right) \end{cases}$$

Assuming that $G(\tau,\sigma)$ is log-linear and given by:

$$G(\tau,\sigma) = \exp\!\left[\frac{1}{2}E(\tau)\sigma^2 + D(\tau)\sigma + C(\tau)\right]$$

where

$$E(0) = 2\eta_3 \quad ; \quad D(0) = 0 \quad ; \quad C(0) = 0$$

leads to:

$$\begin{cases} \dfrac{\partial G}{\partial \tau} = \left[\dfrac{1}{2}E'(\tau)\sigma^2 + D'(\tau)\sigma + C'(\tau)\right] \times G(\tau,\sigma) \\[4mm] \dfrac{\partial G}{\partial \sigma} = \left[E(\tau)\sigma + D(\tau)\right] \times G(\tau,\sigma) \\[4mm] \dfrac{\partial^2 G}{\partial \sigma^2} = E(\tau) \times G(\tau,\sigma) + \left[E(\tau)\sigma + D(\tau)\right]^2 \times G(\tau,\sigma) \end{cases}$$

The PDE satisfied by $G(\tau,\sigma)$ becomes:

$$\frac{1}{2}E'(\tau)\sigma^2 + D'(\tau)\sigma + C'(\tau) = \frac{1}{2}\beta^2\left[E(\tau) + \left(E(\tau)\sigma + D(\tau)\right)^2\right]$$

$$+ (\kappa\theta - \lambda\sigma)\left[E(\tau)\sigma + D(\tau)\right]$$

$$- \left[\eta_2(\tau)\sigma + \eta_1(\tau)\sigma^2\right]$$

After re-arranging it as a polynomial of σ, we deduce the ODEs satisfied by $E(\tau)$:

$$\begin{cases} \dfrac{1}{2}E'(\tau) - \dfrac{1}{2}\beta^2 E^2(\tau) + \lambda E(\tau) + \eta_1(\tau) = 0 \\[4mm] E(0) = 2\eta_3 = \dfrac{\rho}{\beta}\psi \end{cases}$$

by $D(\tau)$:

$$\begin{cases} D'(\tau) - \beta^2 E(\tau)D(\tau) + \lambda D(\tau) - \kappa\theta\, E(\tau) + \eta_2(\tau) = 0 \\[2ex] D(0) = 0 \end{cases}$$

and by $C(\tau)$:

$$\begin{cases} C'(\tau) - \dfrac{1}{2}\beta^2 E(\tau) - \dfrac{1}{2}\beta^2 D^2(\tau) - \kappa\theta\, D(\tau) = 0 \\[3ex] C(0) = 0 \end{cases}$$

The actualized characteristic function is then given by:

$$f(\psi) \equiv E_t^Q\left[\exp\left(-\int_t^T r(s)ds\right)\exp(\psi X_T)\right]$$

$$= e^{-r(T-t)}\exp\left(\psi e^{-\alpha(T-t)}X_t + \frac{\mu}{\alpha}\psi\left(1 - e^{-\alpha(T-t)}\right) - \frac{\rho}{2\beta}\psi e^{-\alpha(T-t)}\sigma_t^2 - \frac{\rho\beta}{2\alpha}\psi\left(1 - e^{-\alpha(T-t)}\right)\right)$$

$$\times \exp\left[\frac{1}{2}E(T-t;\psi)\sigma_t^2 + D(T-t;\psi)\sigma_t + C(T-t;\psi)\right]$$

Appendix B.2: Exact resolution of the ODE satisfied by *E* in the Ornstein-Uhlenbeck framework

The ODEs satisfied by the functions *E* is:

$$\begin{cases} \dfrac{1}{2}E'(\tau) - \dfrac{1}{2}\beta^2 E^2(\tau) + \lambda E(\tau) + \eta_1(\tau) = 0 \\[3ex] E(0) = 2\eta_3 = \dfrac{\rho}{\beta}\psi \end{cases}$$

As detailed in Appendix A.2, we use two transformations before getting the exact solution *E*. Making the first transformation:

$$U(\tau) = \exp\left(-\beta^2\int E(s)ds\right)$$

and the further substitution $V(z) \equiv V(\exp(-\alpha\tau)) \equiv U(\tau)$ lead to:

$$U(\tau) \equiv V(e^{-\alpha\tau})$$
$$= A\exp(-d\alpha\tau)M(a,b,c\psi e^{-\alpha\tau}) + B\exp(-d\alpha\tau)W(a,b,c\psi e^{-\alpha\tau})$$

where again $M(.)$ and $W(.)$ are respectively the *WhittakerM* and the *WhittakerW* functions and:

$$\begin{cases} a = -\dfrac{\rho\alpha + 2\gamma\beta - 2\rho\lambda}{2\alpha\sqrt{\rho^2 - 1}} & ; & b = \dfrac{\lambda}{\alpha} \\[4mm] c = \dfrac{2\beta\sqrt{\rho^2 - 1}}{\alpha} & ; & d = \dfrac{2\lambda - \alpha}{2\alpha} \end{cases}$$

Constants A and B are determined by writing down that $U(0) = 1$ and $U'(0) = -\beta\rho\psi$. We recover the function E by:

$$E(\tau) = -\frac{1}{\beta^2}\frac{U'(\tau)}{U(\tau)}$$

To see that neither D (nor C) could be expressed in a closed-form way, we use a traditional technique to solve linear first-degree ODEs to find:

$$D(\tau) = \frac{\kappa\theta}{\beta^2}\frac{1}{e^{\lambda\tau}U(\tau)}\left(1 - e^{\lambda\tau}U(\tau) + \int_0^\tau \left(\lambda - \rho\beta\psi\, e^{-\alpha s}\right)e^{\lambda s}U(s)\,ds\right)$$

This expression could not be simplified further.

Appendix C: Proof of the well-definiteness of the integrands

For the purpose of integration, we must proof the well-definiteness of the function over the interval of integration, especially around the potential singularities. In particular, we have to value these two integrals:

$$\int_0^{+\infty} \mathrm{Re}\left(\frac{f(1+i\phi)}{i\phi f(1)}K^{-i\phi}\right)d\phi$$

and

$$\int_0^{+\infty} \mathrm{Re}\left(\frac{f(i\phi)}{i\phi f(0)}K^{-i\phi}\right)d\phi$$

where f is a characteristic function, which is of class $C^\infty\left([0,+\infty)\right)$, defined by:

$$f(\psi) = E_t^Q\left(\exp\left\{-\int_t^T r(s)\,ds\right\}\exp\{\psi X_T\}\right)$$

Using a Taylor expansion around $\phi = 0$, the two integrands tend respectively to:

$$\mathrm{Re}\left(\frac{f(i\phi)}{i\phi\,f(0)}K^{-i\phi}\right)\longrightarrow \frac{f'(0)}{f(0)}-\ln(K)=\frac{E_t^{Q}\left(X_T e^{-\int_t^T r(s)ds}\right)}{E_t^{Q}\left(e^{-\int_t^T r(s)ds}\right)}-\ln(K)$$

and

$$\mathrm{Re}\left(\frac{f(1+i\phi)}{i\phi\,f(1)}K^{-i\phi}\right)\longrightarrow \frac{f'(1)}{f(1)}-\ln(K)=\frac{E_t^{Q}\left(e^{-\int_t^T r(s)ds}X_T\exp\{X_T\}\right)}{E_t^{Q}\left(e^{-\int_t^T r(s)ds}\exp\{X_T\}\right)}-\ln(K)$$

Appendix D: Feynman-Kac theorem (Karatzas and Shreve 1991)

Under some regularity assumptions, if we suppose that $F(t,V):[0,T]\times\mathfrak{R}^d\longrightarrow\mathfrak{R}^d$ is of class $C^{1,2}\left([0,T]\times\mathfrak{R}^d\right)$ and satisfies the Cauchy problem:

$$\begin{cases}\dfrac{\partial F}{\partial t}+A_t F+g(t,V)-h(t,V)F(t,V)=0\\[2mm] F(t,V)=k(V)\end{cases}$$

where A_t is the second order differential operator, then $F(t,V)$ is unique and admits the stochastic representation:

$$F(t,V)=E_{t,V}\left[\begin{array}{l}k(V_T)\exp\left\{-\displaystyle\int_t^T h(s,V_s)ds\right\}\\[3mm]+\displaystyle\int_t^T g(\tau,V_\tau)\exp\left\{-\int_t^\tau h(s,V_s)ds\right\}d\tau\end{array}\right]$$

Appendix E: Derivation of the Greeks

Recall that the Call premium is given by:

$$Call(t,T)=f(t,T;1)\,Q_1^{t,T}\left(X_T>\ln(K)\right)-f(t,T;0)\,K\,Q_2^{t,T}\left(X_T>\ln(K)\right)$$

For simplicity, we denote $f(t,T;\psi)$ by $f(\psi)$ and $Q_j^{t,T}\left(X_T>\ln(K)\right)$ by Q_j. We need to compute the following derivatives:

$$\frac{\partial f(\psi)}{\partial X_t} = \psi \, e^{-\alpha(T-t)} f(\psi)$$

$$\frac{\partial\left(\dfrac{f(\psi)}{f(1)}\right)}{\partial X_t} = (\psi - 1)e^{-\alpha(T-t)} \frac{f(\psi)}{f(1)}$$

$$\frac{\partial Q_1}{\partial X_t} = \frac{e^{-\alpha(T-t)}}{\pi} \int_0^{+\infty} \mathrm{Re}\left(\frac{f(1+i\phi)}{f(1)} K^{-i\phi} \right) d\phi$$

$$\frac{\partial Q_2}{\partial X_t} = \frac{e^{-\alpha(T-t)}}{\pi} \int_0^{+\infty} \mathrm{Re}\left(\frac{f(i\phi)}{f(0)} K^{-i\phi} \right) d\phi$$

We thus can write:

$$Delta(t,T) \equiv \frac{\partial Call(t,T)}{\partial e^{X_t}}$$

$$= e^{-X_t} \frac{\partial Call(t,T)}{\partial X_t}$$

$$= e^{-X_t} \left[\frac{\partial f(1)}{\partial X_t} Q_1 + f(1)\frac{\partial Q_1}{\partial X_t} - f(0) K \frac{\partial Q_2}{\partial X_t} \right]$$

By a "formal" change of variable, we can show that:

$$f(1)\frac{\partial Q_1}{\partial X_t} = f(0) K \frac{\partial Q_2}{\partial X_t}$$

and we can deduce the formula for the Delta as given in the main text.

Define the Gamma by:

$$Gamma(t,T) \equiv \frac{\partial Delta(t,T)}{\partial e^{X_t}}$$

Using the same calculus done to derive the Delta leads to:

$$Gamma(t,T) = e^{-X_t} \frac{\partial Delta(t,T)}{\partial X_t}$$

$$= e^{-X_t} \left[\begin{array}{l} -e^{-\alpha(T-t)} e^{-X_t} f(1) Q_1 + e^{-\alpha(T-t)} e^{-X_t} \dfrac{\partial f(1)}{\partial X_t} Q_1 \\[2mm] +e^{-\alpha(T-t)} e^{-X_t} f(1) \dfrac{\partial Q_1}{\partial X_t} \end{array} \right]$$

$$= \left(e^{-\alpha(T-t)} - 1 \right) e^{-X_t} Delta(t,T) + \frac{e^{-2\alpha(T-t)} e^{-2X_t}}{\pi} \int_0^{+\infty} \mathrm{Re}\left(f(1+i\phi) K^{-i\phi} \right) d\phi$$

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For the Vega, consider first the square-root model:

$$Vega(t,T) \equiv \frac{\partial Call(t,T)}{\partial V_t}$$

We compute the following derivatives:

$$\frac{\partial f(\psi)}{\partial V_t} = \left(D(T-t;\psi) - \frac{\rho}{\sigma}\psi\, e^{-\alpha(T-t)} \right) f(\psi)$$

$$\frac{\partial\left(\dfrac{f(\psi)}{f(1)} \right)}{\partial V_t} = \left(D(T-t;\psi) - D(T-t;1) - \frac{\rho}{\sigma}(\psi-1)e^{-\alpha(T-t)} \right)\frac{f(\psi)}{f(1)}$$

$$\frac{\partial Q_1}{\partial V_t} = \frac{1}{\pi}\int_0^{+\infty} \text{Re}\left(\left[\frac{D(T-t;1+i\phi)-D(T-t;1)}{i\phi} - \frac{\rho}{\sigma}e^{-\alpha(T-t)} \right]\frac{f(1+i\phi)}{f(1)}K^{-i\phi} \right) d\phi$$

$$\frac{\partial Q_2}{\partial V_t} = \frac{1}{\pi}\int_0^{+\infty} \text{Re}\left(\left[\frac{D(T-t;i\phi)}{i\phi} - \frac{\rho}{\sigma}e^{-\alpha(T-t)} \right]\frac{f(i\phi)}{f(0)}K^{-i\phi} \right) d\phi$$

We now can write for the Vega:

$$Vega(t,T) = \frac{\partial f(1)}{\partial V_t}Q_1 + f(1)\frac{\partial Q_1}{\partial V_t} - f(0)K\frac{\partial Q_2}{\partial V_t}$$

$$= \left(D(T-t;1) - \frac{\rho}{\sigma}e^{-\alpha(T-t)} \right) f(1)Q_1$$

$$+ \frac{1}{\pi}\int_0^{+\infty} \text{Re}\left(\frac{D(T-t;1+i\phi)-D(T-t;1)}{i\phi}f(1+i\phi)K^{-i\phi} \right) d\phi$$

$$- \frac{K}{\pi}\int_0^{+\infty} \text{Re}\left(\frac{D(T-t;i\phi)}{i\phi}f(i\phi)K^{-i\phi} \right) d\phi$$

For the Ornstein-Uhlenbeck model, define the Vega by:

$$Vega(t,T) \equiv \frac{\partial Call(t,T)}{\partial \sigma_t}$$

The following derivatives are needed:

$$\frac{\partial f(\psi)}{\partial \sigma_t} = \left(E(T-t;\psi)\sigma_t + D(T-t;\psi) - \frac{\rho}{\beta}\psi e^{-\alpha(T-t)}\sigma_t \right) f(\psi)$$

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$$\frac{\partial\left(\dfrac{f(\psi)}{f(1)}\right)}{\partial\sigma_t} = \left(\begin{array}{c} \left[E(T-t;\psi)-E(T-t;1)\right]\sigma_t + D(T-t;\psi) - D(T-t;1) \\ -\dfrac{\rho}{\beta}(\psi-1)e^{-\alpha(T-t)}\sigma_t \end{array}\right)\frac{f(\psi)}{f(1)}$$

$$\frac{\partial Q_1}{\partial\sigma_t} = \frac{1}{\pi}\int_0^{+\infty}\mathrm{Re}\left(\left[\begin{array}{c}\dfrac{E(T-t;1+i\phi)-E(T-t;1)}{i\phi}\sigma_t \\ +\dfrac{D(T-t;1+i\phi)-D(T-t;1)}{i\phi}-\dfrac{\rho}{\beta}e^{-\alpha(T-t)}\sigma_t\end{array}\right]\frac{f(1+i\phi)}{f(1)}K^{-i\phi}\right)d\phi$$

$$\frac{\partial Q_2}{\partial\sigma_t} = \frac{1}{\pi}\int_0^{+\infty}\mathrm{Re}\left(\left[\dfrac{E(T-t;i\phi)}{i\phi}\sigma_t + \dfrac{D(T-t;i\phi)}{i\phi}-\dfrac{\rho}{\beta}e^{-\alpha(T-t)}\sigma_t\right]\frac{f(i\phi)}{f(0)}K^{-i\phi}\right)d\phi$$

The Vega can be written as:

$$Vega(t,T) = \frac{\partial f(1)}{\partial\sigma_t}Q_1 + f(1)\frac{\partial Q_1}{\partial\sigma_t} - f(0)K\frac{\partial Q_2}{\partial\sigma_t}$$

$$=\left(E(T-t;1)\sigma_t + D(T-t;1) - \frac{\rho}{\beta}e^{-\alpha(T-t)}\sigma_t\right)f(1)Q_1$$

$$+\frac{1}{\pi}\int_0^{+\infty}\mathrm{Re}\left(\left[\begin{array}{c}\dfrac{E(T-t;1+i\phi)-E(T-t;1)}{i\phi}\sigma_t \\ +\dfrac{D(T-t;1+i\phi)-D(T-t;1)}{i\phi}\end{array}\right]f(1+i\phi)K^{-i\phi}\right)d\phi$$

$$-\frac{K}{\pi}\int_0^{+\infty}\mathrm{Re}\left(\left[\dfrac{E(T-t;i\phi)}{i\phi}\sigma_t + \dfrac{D(T-t;i\phi)}{i\phi}\right]f(i\phi)K^{-i\phi}\right)d\phi$$

Notes

[1] The square-root diffusion generalizes the traditional mean reverting process by not allowing the state variable to be negative.

[2] In this setting, only options on the true underlying asset e^X are considered, but the methodology can easily handle options on the state variable X itself.

[3] This can be seen as an "actualized moment generating function" of the true underlying asset e^X.

[4] The ODEs (26) and (27) have analytic solutions that involve the Whittaker functions. However, they need much more time to be computed for large values of ϕ than solved numerically for the same order of accuracy. The series expansion for the Fong and Vasicek (1992) discount bond price in Selby and Strickland (1995) is efficient because this is equivalent to ϕ being always equal to 1.

[5] Heston's (1993) model is one of the models presented in Zhu (2000).

[6] This asymptotic price can be computed with a Monte Carlo simulation.