A Theoretical Extension of the Consumption-based CAPM Model

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Abstract

We extend the Consumption-based CAPM (C-CAPM) model to representative agents with different risk attitudes. We use the concept of expectation dependence and show that for a risk averse representative agent, it is the first-degree expectation dependence rather than the covariance that determines C-CAPM's riskiness. We extend the assumption of risk aversion to prudence and provide an additional dependence condition to obtain the values of asset price and equity premium. Results are generalized to higher-degree risk changes and higher- order risk averse representative agents, and are linked to the equity premium puzzle.

Key words: Consumption-based CAPM; Risk premium; Equity premium puzzle; Expectation dependence; Ross risk aversion.

JEL classification: D51; D80; G12

1 Introduction

The consumption-based capital asset pricing model (C-CAPM), developed in Rubinstein (1976), Lucas (1978), Breeden (1979) and Grossman and Shiller (1981), relates the risk premium on each asset to the covariance between the asset return and the intertemporal marginal rate of substitution of a decision maker. The most important comparative static results for C-CAPM is how an asset's price or equity premium changes as the quantity of risk and the price of risk change. The results of comparative statics analysis thus form the basis for much of our understanding of the sources of changes in consumption (macroeconomic) risk and risk aversion that drive asset prices and equity premia.

The two objectives of this study are to propose a new theoretical framework for C-CAPM and to extend its comparative statics. We use general utility functions and probability distributions to investigate C-CAPM. Our model provides insight into the basic concepts that determine asset prices and equity premia.

The C-CAPM pricing rule is sometimes interpreted as implying that the price of an asset with a random payoff falls short of its expected payoff if and only if the random payoff positively correlates with consumption. Liu and Wang (2007) show that this interpretation of C-CAPM is not generally correct by presenting a counterexample. We use more powerful statistical tools to obtain the appropriate dependence between asset payoff and consumption. We first discuss the concept of expectation dependence developed by Wright (1987) and Li (2011). We show that, for a risk averse representative agent, it is the first-degree expectation dependence between the asset's payoff and consumption rather than the covariance that determines C-CAPM's riskiness. Our result also reinterprets the covariance between an asset's payoff and the marginal utility of consumption in terms of the expectation dependence between the asset's payoff and consumption itself. We extend the assumption of risk aversion to prudence and provide another dependence condition. Finally, we interpret C-CAPM in a general setting: for the i^{th}-degree risk averse representative agent,[1] with $i = 2, .., N+1$, it is the N^{th}-order expectation dependence that determines C-CAPM's riskiness.

[1]Risk aversion in the traditional sense of a concave utility function is indicated by $i = 2$, whereas $i = 3$ corresponds to downside risk aversion in the sense of Menezes, Geiss and Tressler (1980). i^{th}-degree risk aversion is equivalent to preferences satisfying risk apportionment of order i. See Ekern (1980) and Eeckhoudt and Schlesinger (2006) for more discussions.

Our contribution is linked to the recent literature on higher-order risk preferences and higher-order moments and comoments in finance developed by Harvey and Siddique (2000), Dittmar (2002), Mitton and Vorkink (2007), Chabi-Yo et al. (2007), and Martellini and Ziermann (2010). We provide a theoretical foundation for the pricing kernel model based on higher comoments than the covariance by suggesting a more general definition of dependence between consumption and asset payoff, and propose a general pricing formula not restricted to specific utility functions.

Our study also relates to Gollier and Schlesinger (2002) who examine asset prices in a representative-agent model of general equilibrium with two differences. First, we study asset price and equity premium driven by macroeconomic risk as in the traditional C-CAPM model while Gollier and Schlesinger's model considers the relationship between the riskiness of the market portfolio and its expected return. Second, Gollier and Schlesinger's model is a one period model whereas our results rest on a two-period framework.

Finally, our study extends the literature that examines the effects of higher-degree risk changes on the economy. Eeckhoudt and Schlesinger (2006) investigate necessary and sufficient conditions on preferences for a higher-degree change in risk to increase saving. Our study provides necessary and sufficient conditions on preferences for a higher-degree change in risk to set asset prices, and sufficient conditions on preferences for a higher-degree change in risk to set equity premia.

The paper proceeds as follows. Section 2 introduces several concepts of dependence. Section 3 provides a reinterpretation of C-CAPM for risk averse representative agents. Section 4 extends the results of Section 3 to prudent and higher-order risk averse agents respectively. Section 5 discusses the results in relation to local indexes of risk aversion and higher-order moments and comoments. Section 6 interprets the results in terms of the equity premium puzzle and concludes the paper.

2 Concepts of dependence

The concept of correlation coined by Galton (1886) had served as the only measure of dependence during the first 70 years of the 20th century. However correlation is too weak to obtain meaningful conclusions in many economic and financial applications. For example, covariance is a poor tool for describing dependence for non-normal distributions. Since Lehmann's introduction of the concept of quadrant dependence in 1966, stronger definitions of dependence have received much attention in the statistical literature[2].

Suppose $\tilde{x} \times \tilde{y} \in R \times R$ are two continuous random variables. Let $F(x, y)$ denote the joint and $F_x(x)$ and $F_y(y)$ the marginal distributions of $\tilde{x}$ and $\tilde{y}$. Lehmann (1966) introduces the following concept to investigate positive dependence.

Definition 2.1 *(Lehmann, 1966) $(\tilde{x}, \tilde{y})$ is positively quadrant dependent, written as $PQD(\tilde{x}, \tilde{y})$, if*

$$F(x, y) \geq F_x(x)F_y(y) \qquad for\ all\ (x, y) \in R \times R. \tag{1}$$

The above inequality can be rewritten as

$$F_x(x|\tilde{y} \leq y) \geq F_x(x) \tag{2}$$

and Lehmann provides the following interpretation of definition (2.1): "knowledge of $\tilde{y}$ being small increases the probability of $\tilde{x}$ being small." PQD is useful to model dependent risks because it can take into account the simultaneous downside (upside) evolution of risks. The marginal and the conditional CDFs can be changed simultaneously.[3]

Wright (1987) introduced the concept of expectation dependence in the economics literature. The following definition, uses a weaker definition of dependence than PQD.

Definition 2.2 *If*

$$FED(\tilde{x}|y) = [E\tilde{x} - E(\tilde{x}|\tilde{y} \leq y)] \geq 0\ for\ all\ y \in R, \tag{3}$$

$$then\ \tilde{x}\ is\ positive\ first-degree\ expectation\ dependent\ on\ \tilde{y}.$$

The family of all distributions F satisfying (3) will be denoted by $\mathcal{F}_1$. Similarly, $\tilde{x}$ is negative first-degree expectation dependent on $\tilde{y}$ if (3) holds with the inequality sign reversed. The totality of negative first-degree expectation dependent distributions will be denoted by $\mathcal{G}_1$.

Wright (1987, page 113) interprets negative first-degree expectation dependence as follows: "when we discover $\tilde{y}$ is small, in the precise sense that we are given the truncation $\tilde{y} \leq y$, our expectation of $\tilde{x}$ is revised upward." First-degree expectation dependence is a stronger definition of dependence than correlation, but a weaker definition than quadrant dependence.

[2]For surveys of the literature, we refer to Joe (1997), Mari and Kotz (2001) and Embrechts (2009).

[3]Portfolio selection problems with positive quadrant dependency have been explored by Pellerey and Semeraro (2005) and Dachraoui and Dionne (2007), among others. Pellerey and Semeraro (2005) assert that a large subset of the elliptical distributions class is PQD. For examples of distributions, see Joe (1997).

Therefore, many bivariate random variables being positive (negative) quadrant dependent are also first-degree expectation dependent. But positive (negative) correlated random variables are not necessary positive (negative) first-degree expectation dependent. We provide a simple example of two random variables that are positively correlated but not positive first-degree expectation dependent.

Example Let $\tilde{x}$ be normally distributed with $E\tilde{x} = \mu > 0$ and $var(\tilde{x}) = \sigma^2$. Let $\tilde{y} = \tilde{x}^2$. Since $E\tilde{x}^2 = \mu^2 + \sigma^2$ and $E\tilde{x}^3 = \mu^3 + 3\mu\sigma^2$, then

$$\begin{aligned} cov(\tilde{x}, \tilde{y}) &= E\tilde{x}\tilde{y} - E\tilde{x}E\tilde{y} \\ &= E\tilde{x}^3 - E\tilde{x}E\tilde{x}^2 \\ &= \mu^3 + 3\mu\sigma^2 - \mu(\mu^2 + \sigma^2) = 2\mu\sigma^2 > 0. \end{aligned} \tag{4}$$

By definition,

$$\begin{aligned} FED(\tilde{y}| - \sqrt{\mu^2 + \sigma^2}) &= E\tilde{x}^2 - E(\tilde{x}^2 | \tilde{x} \leq -\sqrt{\mu^2 + \sigma^2}) \\ &= \mu^2 + \sigma^2 - E(\tilde{x}^2 | \tilde{x} \leq -\sqrt{\mu^2 + \sigma^2}) < 0, \end{aligned} \tag{5}$$

and we obtain $(\tilde{y}, \tilde{x}) \notin \mathcal{F}_1$.

Note that the expectation dependence concept is not symmetric (for more discussion, see Wright, 1987). Li (2011) proposes the following weaker definition of dependence:

Definition 2.3 *If*

$$\begin{aligned} SED(\tilde{x}|y) &= \int_{-\infty}^{y} [E\tilde{x} - E(\tilde{x}|\tilde{y} \leq t)] F_y(t) dt \\ &= \int_{-\infty}^{y} FED(\tilde{x}|t) F_y(t) dt \geq 0 \ for\ all\ y, \end{aligned} \tag{6}$$

then $\tilde{x}$ is positive second-degree expectation dependent on $\tilde{y}$.

The family of all distributions F satisfying (6) will be denoted by $\mathcal{F}_2$. Similarly, $\tilde{x}$ is negative second-degree expectation dependent on $\tilde{y}$ if (6) holds with the inequality sign reversed, and the totality of negative second-degree expectation dependent distributions will be denoted by $\mathcal{G}_2$.

It is obvious that $\mathcal{F}_1 \subseteq \mathcal{F}_2$ and $\mathcal{G}_1 \subseteq \mathcal{G}_2$ but the converse is not true. Because $\tilde{x}$ and $\tilde{y}$ are positively correlated when (see Lehmann 1966, lemma 2)

$$cov(\tilde{x}, \tilde{y}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} [F(x, y) - F_x(x) F_y(y)] dx dy = \int_{-\infty}^{+\infty} FED(\tilde{x}|t) F_y(t) dt \geq 0, \tag{7}$$

again we see that $cov(\tilde{x}, \tilde{y}) \geq 0$ is only a necessary condition for $(\tilde{x}, \tilde{y}) \in \mathcal{F}_2$ but the converse is not true. Comparing (6) and (7), we know that $cov(\tilde{x}, \tilde{y})$ is the 2nd central cross moment of $\tilde{x}$ and $\tilde{y}$ while $SED(\tilde{x}|y)$ is related to 2nd central cross lower partial moment of $\tilde{x}$ and $\tilde{y}$ which can be explained as a measure of downside risk computed as the average of the squared deviations below a target.

For our purpose of extending the C-CAPM model, comparative expectation dependence has to be defined. We propose the following definition to quantify comparative expectation dependence.

Definition 2.4 *Distribution $F(x, y)$ is more first-degree expectation dependent than $H(x, y)$ if and only if $FED_F(\tilde{x}|y)F_y(y) \geq FED_H(\tilde{x}|y)H_y(y)$ for all y. Distribution $F(x, y)$ is more second order expectation dependent than $H(x, y)$, if $FED_F(\tilde{x}) \geq FED_H(\tilde{x})$, and*

$$SED_F(\tilde{x}|y) \geq SED_H(\tilde{x}|y) \ for \ all \ y. \tag{8}$$

Sibuya (1960) introduces the concept of dependence function Ω_F:

$$\Omega_F = \frac{F(x, y)}{F_x(x)F_y(y)}. \tag{9}$$

Since $\log(\Omega_F) = \log(F(x, y)) - \log(F_x(x)F_y(y))$, "more first-degree expectation dependent" compares the difference between $F(x, y)$ and $F_x(x)F_y(y)$ whereas Sibuya's definition compares the difference between $\log(F(x, y))$ and $\log(F_x(x)F_y(y))$.

3 C-CAPM for a risk averse representative agent

3.1 Consumption-based asset pricing model

Suppose that an investor can freely buy or sell an asset with random payoff $\tilde{x}_{t+1}$ at a price p_t. The investor's preference can be represented by a utility function $u(.)$. We assume that all derivatives for $u(.)$ exist. Denote ξ as the amount of the asset the investor chooses to buy. Then, his decision problem is to

$$\begin{aligned} & \max_{\xi} u(c_t) + \beta E_t[u(\tilde{c}_{t+1})] \qquad (10) \\ s.t. \quad & c_t = e_t - p_t\xi \\ & \tilde{c}_{t+1} = e_{t+1} + \tilde{x}_{t+1}\xi, \end{aligned}$$

where e_t and e_{t+1} are the original consumption levels, β is the subjective discount factor, c_t is the consumption in period t, and $\tilde{c}_{t+1}$ is the consumption in period $t+1$.

From the first order condition of this problem, we can obtain the well-known consumption-based asset pricing model which can be expressed by the following two equations (see e.g. Cochrane 2005, page 13-14)[4]

$$p_t = \frac{E_t \tilde{x}_{t+1}}{R^f} + \beta \frac{cov_t[u'(\tilde{c}_{t+1}), \tilde{x}_{t+1}]}{u'(c_t)}, \tag{11}$$

and

$$E_t \tilde{R}_{t+1} - R^f = -\frac{cov_t[u'(\tilde{c}_{t+1}), \tilde{R}_{t+1}]}{E_t u'(\tilde{c}_{t+1})} \tag{12}$$

where gross return $\tilde{R}_{t+1} = \frac{\tilde{x}_{t+1}}{P_t}$ in period $t+1$, R^f is the gross return of the risk-free asset, $u'(\cdot)$ is the marginal utility function, $E\tilde{R}_{t+1} - R^f$ is the asset's risk premium. As for expectation dependence, $cov_t[u'(\tilde{c}_{t+1}), \tilde{x}_{t+1}]$ and $cov_t[u'(\tilde{c}_{t+1}), \tilde{R}_{t+1}]$ are not symmetric.

The first term on the right-hand side of (11) is the standard discounted present-value formula. This is the asset's price for a risk-neutral representative agent or for a representative agent when asset payoff and consumption are independent. The second term is a risk aversion adjustment. (11) states that an asset with random future payoff $\tilde{x}_{t+1}$ is worth less than its expected payoff discounted at the risk-free rate if and only if $cov[u'(\tilde{c}_{t+1}), \tilde{x}_{t+1}] \leq 0$. (12) shows that an asset has an expected return equal to the risk-free rate plus a risk adjustment under risk aversion.

When the representative agent's utility function is the power function, $u(c_t) = \frac{c_t^{1-\gamma}-1}{1-\gamma}$ where γ is the coefficient of relative risk aversion and $\tilde{c}_{t+1}$ and $\tilde{x}_{t+1}$ are conditional lognormally distributed, (12) becomes (Campbell 2003, page 821)

$$E_t \tilde{r}_{t+1} - r^f + \frac{var_t(\tilde{r}_{t+1})}{2} = \gamma cov_t(\log \tilde{c}_{t+1}, \tilde{r}_{t+1}), \tag{13}$$

where $\tilde{r}_{t+1} = \log(1 + \tilde{R}_{t+1})$ and $r^f = \log(1 + R^f)$.

(13) states that the log risk premium is equal to the product of the coefficient of relative risk aversion and the covariance of the log asset return with consumption growth. We now provide a generalization of these results.

From Theorem 1 in Cuadras (2002), we know that covariance can always be written as

$$cov_t[u'(\tilde{c}_{t+1}), \tilde{x}_{t+1}] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} [F(c_{t+1}, x_{t+1}) - F_{c_{t+1}}(c_{t+1}) F_{x_{t+1}}(x_{t+1})] u''(c_{t+1}) dx_{t+1} dc_{t+1}. \tag{14}$$

[4]Equations (11) and (12) can also be obtained in a muti-period dynamic framework from Euler equations. For more details, see Constantinides and Duffie (1996).

Because we can write

$$\int_{-\infty}^{+\infty}[F_{x_{t+1}}(x_{t+1}|\tilde{c}_{t+1} \leq c_{t+1}) - F_{x_{t+1}}(x_{t+1})]dx_{t+1} = E\tilde{x}_{t+1} - E(\tilde{x}_{t+1}|\tilde{c}_{t+1} \leq c_{t+1}), \qquad (15)$$

(see, e.g., Tesfatsion (1976), Lemma 1), hence, we have

$$\begin{aligned} & cov_t[u'(\tilde{c}_{t+1}), \tilde{x}_{t+1}] && (16) \\ = & \int_{-\infty}^{+\infty}[E\tilde{x}_{t+1} - E(\tilde{x}_{t+1}|\tilde{c}_{t+1} \leq c_{t+1})]F_{c_{t+1}}(c_{t+1})u''(c_{t+1})dc_{t+1} && (by\ (15)) \\ = & \int_{-\infty}^{+\infty} FED(\tilde{x}_{t+1}|c_{t+1})u''(c_{t+1})F_{c_{t+1}}(c_{t+1})dc_{t+1}. \end{aligned}$$

Using (16), (11) can be rewritten as

$$\begin{aligned} p_t &= \underbrace{\frac{E_t\tilde{x}_{t+1}}{R^f}}_{discounted\ present\ value\ effect} \underbrace{- \beta \int_{-\infty}^{+\infty} FED(\tilde{x}_{t+1}|c_{t+1})F_{c_{t+1}}(c_{t+1})[-\frac{u''(c_{t+1})}{u'(c_t)}]dc_{t+1}}_{first-degree\ expectation\ dependence\ effect} \qquad (17) \\ &= \frac{E_t\tilde{x}_{t+1}}{R^f} - \beta \int_{-\infty}^{+\infty} FED(\tilde{x}_{t+1}|c_{t+1})F_{c_{t+1}}(c_{t+1})AR(c_{t+1})MRS_{c_{t+1},c_t}dc_{t+1}, \end{aligned}$$

where $AR(x) = -\frac{u''(x)}{u'(x)}$ is the Arrow-Pratt absolute risk aversion coefficient, and $MRS_{x,y} = \frac{u'(x)}{u'(y)}$ is the marginal rate of substitution between x and y.[5]

We can also rewrite (12) as

$$E_t\tilde{R}_{t+1} - R^f = \int_{-\infty}^{+\infty} \underbrace{FED(\tilde{R}_{t+1}|c_{t+1})F_{c_{t+1}}(c_{t+1})}_{consumption\ risk\ effect} \underbrace{[-\frac{u''(c_{t+1})}{E_tu'(\tilde{c}_{t+1})}]}_{price\ of\ risk\ effect} dc_{t+1} \qquad (18)$$

Because $R^f = \frac{1}{\beta}\frac{u'(c_t)}{E_tu'(\tilde{c}_{t+1})}$ (see e.g. Cochrane 2005, page 11), we also have

$$E_t\tilde{R}_{t+1} - R^f = \beta R^f \int_{-\infty}^{+\infty} FED(\tilde{R}_{t+1}|c_{t+1})F_{c_{t+1}}(c_{t+1})AR(c_{t+1})MRS_{c_{t+1},c_t}dc_{t+1} \qquad (19)$$

(17) shows that an asset's price involves two terms. The effect, measured by the first term on the right-hand side of (17), is the "discounted present value effect." This effect depends on the expected return of the asset and the risk-free interest rate. The sign of the discounted present value effect is the same as the sign of the expected return. This term captures the "direct" effect of the discounted expected return, which characterizes the asset's price for a risk-neutral representative agent.

The second term on the right-hand side of (17) is called "first-degree expectation dependence effect." This term involves the subjective discount factor, the expectation dependence

[5] Andersen et al. (2011) propose a multi-attribute risk aversion model that allows one to separate the intertemporal risk aversion coefficient into the risk aversion coefficient and the MRS

between the random payoff and consumption, the Arrow-Pratt risk aversion coefficient and the intertemporal marginal rate of substitution. The sign of the first-degree expectation dependence indicates whether the movements on consumption tend to reinforce (positive first-degree expectation dependence) or to counteract (negative first-degree expectation dependence) the movements on an asset's payoff.

(18) states that the expected excess return on any risky asset over the risk-free interest rate can be explained as the sum of the quantity of consumption risk times the price of this risk. The quantity of consumption risk is measured by the first-degree expectation dependence of the excess stock return with consumption, while the price of risk is the Arrow-Pratt risk aversion coefficient times the intertemporal marginal rate of substitution.

We obtain the following proposition from (17) and (18).

Proposition 3.1 *Suppose $F(x_{t+1}, c_{t+1})$ and $F(R_{t+1}, c_{t+1})$ are continuous, then the following statements hold:*

(i) $p_t \leq \frac{E_t\tilde{x}_{t+1}}{R^f}$ for any risk averse representative agent ($u'' \leq 0$) if and only if $(\tilde{x}_{t+1}, \tilde{c}_{t+1}) \in \mathcal{F}_1$;

(ii) $p_t \geq \frac{E_t\tilde{x}_{t+1}}{R^f}$ for any risk averse representative agent ($u'' \leq 0$) if and only if $(\tilde{x}_{t+1}, \tilde{c}_{t+1}) \in \mathcal{G}_1$;

(iii) $E_t\tilde{R}_{t+1} \geq R^f$ for any risk averse representative agent ($u'' \leq 0$) if and only if $(\tilde{R}_{t+1}, \tilde{c}_{t+1}) \in \mathcal{F}_1$;

(iv) $E_t\tilde{R}_{t+1} \leq R^f$ for any risk averse representative agent ($u'' \leq 0$) if and only if $(\tilde{R}_{t+1}, \tilde{c}_{t+1}) \in \mathcal{G}_1$.

Proof See Appendix A.

Proposition 3.1 states that, for a risk averse representative agent, an asset's price is lowered (or equity premium is positive) if and only if its payoff is positively first-degree expectation dependent with consumption. Conversely, an asset's price is raised (or equity premium is negative) if and only if its payoff is negatively first-degree expectation dependent with consumption. Therefore, for a risk averse representative agent, it is the first-degree expectation dependence rather than the covariance that determines its riskiness. Because $(\tilde{x}_{t+1}, \tilde{c}_{t+1}) \in \mathcal{F}_1(\mathcal{G}_1) \Rightarrow cov_t(\tilde{x}_{t+1}, \tilde{c}_{t+1}) \geq 0 (\leq 0)$ and the converse is not true, we conclude that a positive (negative) covariance is a necessary but not sufficient condition for a risk averse agent paying a lower (higher) asset price (or having a positive (negative) equity premium).

3.2 Comparative risk aversion

The assumption of risk aversion has long been a cornerstone of modern economics and finance. Ross (1981) provides the following strong measure for comparative risk reversion:

Definition 3.2 *(Ross 1981) u is more Ross risk averse than v if and only if there exists $\lambda > 0$ such that for all x, y*

$$\frac{u''(x)}{v''(x)} \geq \lambda \geq \frac{u'(y)}{v'(y)}. \tag{20}$$

More risk averse in the sense of Ross guarantees that the more risk averse decision-maker is willing to pay more to benefit from a mean preserving contraction.

Our important comparative statics question is the following: Under which condition does a change in the representative agent's risk preferences reduce the asset price? To answer this question let us consider a change in the utility function from u to v. From (17), for agent v, we have

$$p_t = \frac{E_t\tilde{x}_{t+1}}{R^f} - \beta \int_{-\infty}^{+\infty} FED(\tilde{x}_{t+1}|c_{t+1})F_{C_{t+1}}(c_{t+1})[-\frac{v''(c_{t+1})}{v'(c_t)}]dc_{t+1}. \tag{21}$$

Intuition suggests that if asset return and consumption are positive dependent and agent v is more risk averse than agent u, then agent v should have a larger risk premium than agent u. This intuition can be reinforced by Ross risk aversion and first-degree expectation dependence, as stated in the following proposition.

Proposition 3.3 *Let p_t^u and p_t^v denote the asset's prices corresponding to u and v respectively. Suppose u', v', u'' and v'' are continuous, then the following statements hold:*

(i) $p_t^u \geq p_t^v$ for all $(\tilde{x}_{t+1}, \tilde{c}_{t+1}) \in \mathcal{F}_1$ if and only if v is more Ross risk averse than u;

(ii) $p_t^u \geq p_t^v$ for all $(\tilde{x}_{t+1}, \tilde{c}_{t+1}) \in \mathcal{G}_1$ if and only if u is more Ross risk averse than v;

Proof See Appendix A.

Proposition 3.3 indicates that, when an asset's payoff is first-degree positive (negative) expectation dependent on consumption, an increase in risk aversion in the sense of Ross decreases (increases) the asset price.

The results of Proposition 3.3 cannot be obtained with the Arrow-Pratt relative risk aversion measure. We consider the convenient power utility form $u(c) = \frac{c^{1-\gamma_u}-1}{1-\gamma_u}$ and $v(c) = \frac{c^{1-\gamma_v}-1}{1-\gamma_v}$. γ_u and γ_v are u and v's relative risk aversion coefficients respectively. Intuition would suggest that, when an asset's payoff and consumption are positively dependent, $\gamma_u \geq \gamma_v$ implies that

$p_t^u \leq p_t^v$. However, the following counter example shows that, in the range of acceptable values of parameters, a lower Arrow-Pratt relative risk aversion coefficient is neither a necessary nor a sufficient condition to obtain higher price when asset's payoff and consumption are first-degree positive expectation dependent[6].

Counter Example Suppose $u(c) = \frac{c^{1-\gamma}-1}{1-\gamma}$, $\tilde{c}_{t+1} \in [1,3]$ almost surely and $(\tilde{x}_{t+1}, \tilde{c}_{t+1}) \in \mathcal{F}_1$ (note that in this case $cov_t(\tilde{x}_{t+1}, \tilde{c}_{t+1}) \geq 0$), from (18) we obtain

$$\begin{aligned} p_t &= \frac{E_t\tilde{x}_{t+1}}{R^f} - \beta\int_{-\infty}^{+\infty} FED(\tilde{x}_{t+1}|c_{t+1})F_{c_{t+1}}(c_{t+1})[-\frac{u''(c_{t+1})}{u'(c_t)}]dc_{t+1} \\ &= \frac{E_t\tilde{x}_{t+1}}{R^f} - \beta\int_{-\infty}^{+\infty} FED(\tilde{x}_{t+1}|c_{t+1})F_{c_{t+1}}(c_{t+1})\frac{\gamma c_{t+1}^{-\gamma-1}}{E_t\tilde{c}_{t+1}^{-\gamma}}dc_{t+1}, \end{aligned} \tag{22}$$

hence, when an asset's payoff and consumption are positively first-degree expectation dependent, $\frac{dp_t}{d\gamma} \leq 0$ if and only if $d\frac{\gamma c_{t+1}^{-\gamma-1}}{E_t\tilde{c}_{t+1}^{-\gamma}}/d\gamma \geq 0$. We now show that $d\frac{\gamma c_{t+1}^{-\gamma-1}}{E_t\tilde{c}_{t+1}^{-\gamma}}/d\gamma \geq 0$ is not always verified because it contains the variations of the marginal rate of substitution. Since

$$d\frac{\gamma c_{t+1}^{-\gamma-1}}{E_t\tilde{c}_{t+1}^{-\gamma}}/d\gamma = \frac{c_{t+1}^{-\gamma-1}}{(E_t\tilde{c}_{t+1}^{-\gamma})^2}\{[1-\gamma(\gamma+1)c_{t+1}^{-1}]E_t\tilde{c}_{t+1}^{-\gamma} + \gamma^2 E_t\tilde{c}_{t+1}^{-\gamma-1}\}, \tag{23}$$

we obtain $\frac{dp_t}{d\gamma} \leq 0$ if and only if $[1-\gamma(\gamma+1)c_{t+1}^{-1}]E_t\tilde{c}_{t+1}^{-\gamma} + \gamma^2 E_t\tilde{c}_{t+1}^{-\gamma-1} \geq 0$. Because

$$\begin{aligned} & [1-\gamma(\gamma+1)c_{t+1}^{-1}]E_t\tilde{c}_{t+1}^{-\gamma} + \gamma^2 E_t\tilde{c}_{t+1}^{-\gamma-1} \\ \leq\ & [1-\frac{1}{3}\gamma(\gamma+1)]E_t\tilde{c}_{t+1}^{-\gamma} + \gamma^2 E_t\tilde{c}_{t+1}^{-\gamma-1} \quad (\textit{since } \tilde{c}_{t+1} \leq 3 \textit{ almost surely}), \end{aligned} \tag{24}$$

then for $\gamma = 2$ and $\tilde{c}_{t+1}$ such that $E_t\tilde{c}_{t+1}^{-\gamma} = \frac{1}{5}$ and $E_t\tilde{c}_{t+1}^{-\gamma-1} = \frac{1}{21}$, we have

$$[1-\frac{1}{3}\gamma(\gamma+1)]E_t\tilde{c}_{t+1}^{-\gamma} + \gamma^2 E_t\tilde{c}_{t+1}^{-\gamma-1} = -\frac{1}{5} + \frac{4}{21} < 0. \tag{25}$$

Therefore, a higher Arrow-Pratt relative risk aversion coefficient is neither a necessary nor a sufficient condition to obtain a lower price.

3.3 Changes in joint distributions

The question dual to the change in risk aversion examined above is as follows: Under which condition does a change in the joint distribution of random payoff and consumption increase the asset's price? We may also ask the same question for the risk premium by using the joint distribution of an asset's gross return and consumption. To address these questions, let us denote

[6]When $\tilde{x}_{t+1} = \tilde{c}_{t+1}$, the asset price is always monotone in the Arrow-Pratt measure of risk aversion (Gollier 2001, p336-337).

$E_t^H$and FED_H as the expectation and first order expectation dependency under distribution $H(x,y)$. Let p_t^F and p_t^H denote the corresponding prices under distributions $F(x,y)$ and $H(x,y)$ respectively. From (17), we have

$$p_t^H = \frac{E_t^H \tilde{x}_{t+1}}{R^f} - \beta \int_{-\infty}^{+\infty} FED_H(\tilde{x}_{t+1}|c_{t+1}) H_{C_{t+1}}(c_{t+1})[-\frac{u''(c_{t+1})}{u'(c_t)}]dc_{t+1}. \tag{26}$$

From (17) and (26), we obtain the following result.

Proposition 3.4 *(i) Suppose $F(x_{t+1}, c_{t+1})$ is continuous and $E_t^F \tilde{x}_{t+1} = E_t^H \tilde{x}_{t+1}$, then $p_t^F \leq p_t^H$ for all risk averse representative agents if and only if $F(x_{t+1}, c_{t+1})$ is more first-degree expectation dependent than $H(x_{t+1}, c_{t+1})$;*

(ii)Suppose $F(R_{t+1}, c_{t+1})$ is continuous, then for all risk averse representative agents, $F(R_{t+1}, c_{t+1})$ is more first-degree expectation dependent than $H(R_{t+1}, c_{t+1})$ if and only if the risk premium under $F(R_{t+1}, c_{t+1})$ is greater than under $H(R_{t+1}, c_{t+1})$.

Proof See Appendix A.

Proposition 3.4 shows that a pure increase in first-degree expectation dependence represents an increase in asset riskiness for all risk averse investors. The next corollary considers a simultaneous variation in expected return and dependence.

Corollary 3.5 *For all risk averse representative agents, $E_t^F \tilde{x}_{t+1} \leq E_t^H \tilde{x}_{t+1}$ and $F(x_{t+1}, c_{t+1})$ is more first-degree expectation dependent than $H(x_{t+1}, c_{t+1})$ imply $p_t^F \leq p_t^H$.*

Proof The sufficient conditions are directly obtained from (17) and (26).

Corollary 3.5 states that, for all risk averse representative agents, a decrease in the expected return and an increase in the first-degree expectation dependence between return and consumption will decrease the asset's price. Again, the key available concept for prediction is comparative first-degree expectation dependence.

4 C-CAPM for a higher-order risk averse representative agent

4.1 C-CAPM for a risk averse and prudent representative agent

The concept of prudence and its relationship to precautionary savings was introduced by Kimball (1990). Since then, prudence has become a common and accepted assumption in the economics literature (Gollier 2001). All prudent agents dislike any increase in downside risk in the sense of Menezes et al. (1980) (see also Chiu, 2005.). Deck and Schlesinger (2010) conduct a laboratory experiment to determine whether preferences are prudent and show behavioral evidence for prudence. In this section, we will demonstrate that we can get additional dependence conditions for asset price and equity premium than first-degree expectation dependence, when the representative agent is risk averse and prudent.

We can integrate the right-hand term of (16) by parts and obtain:

$$\begin{aligned}
cov_t[u'(\tilde{c}_{t+1}),\tilde{x}_{t+1}] &= \int_{-\infty}^{+\infty} FED(\tilde{x}_{t+1}|c_{t+1})u''(c_{t+1})F_{c_{t+1}}(c_{t+1})dc_{t+1} \\
&= \int_{-\infty}^{+\infty} u''(c_{t+1})d(\int_{-\infty}^{c_{t+1}}[E\tilde{x}_{t+1}-E(\tilde{x}_{t+1}|\tilde{c}_{t+1}\le s)]F_{c_{t+1}}(s)ds) \\
&= u''(c_{t+1})\int_{-\infty}^{c_{t+1}}[E\tilde{x}_{t+1}-E(\tilde{x}_{t+1}|\tilde{c}_{t+1}\le s)]F_{c_{t+1}}(s)ds|_{-\infty}^{+\infty} \\
&\quad -\int_{-\infty}^{+\infty}\int_{-\infty}^{c_{t+1}}[E\tilde{x}_{t+1}-E(\tilde{x}_{t+1}|\tilde{c}_{t+1}\le s)]F_{c_{t+1}}(s)ds u'''(c_{t+1})dc_{t+1} \\
&= u''(+\infty)\int_{-\infty}^{+\infty}[E\tilde{x}_{t+1}-E(\tilde{x}_{t+1}|\tilde{c}_{t+1}\le s)]F_{c_{t+1}}(s)ds \\
&\quad -\int_{-\infty}^{+\infty}\int_{-\infty}^{c_{t+1}}[E\tilde{x}_{t+1}-E(\tilde{x}_{t+1}|\tilde{c}_{t+1}\le s)]F_{c_{t+1}}(s)ds u'''(c_{t+1})dc_{t+1} \\
&= u''(+\infty)cov_t(\tilde{x}_{t+1},\tilde{c}_{t+1})-\int_{-\infty}^{+\infty} SED(\tilde{x}_{t+1}|c_{t+1})u'''(c_{t+1})dc_{t+1}.
\end{aligned} \tag{27}$$

From equation (7), we know that a positive SED implies a positive $cov(\tilde{x}_{t+1},\tilde{c}_{t+1})$ but the converse is not true. Hence, we have from (27) that $cov_t(\tilde{x}_{t+1},\tilde{c}_{t+1})\ge 0$ is a necessary but not sufficient condition for $cov_t[u'(\tilde{c}_{t+1}),\tilde{x}_{t+1}]\le 0$ for all $u''\le 0$ and $u'''\ge 0$. With a positive SED function, prudence is also necessary.

(11) and (12) can be rewritten as:

$$\begin{aligned}
p_t &= \underbrace{\frac{E_t\tilde{x}_{t+1}}{R^f}}_{discounted\ present\ value\ effect} \underbrace{-\beta cov_t(\tilde{x}_{t+1},\tilde{c}_{t+1})[-\frac{u''(+\infty)}{u'(c_t)}]}_{covariance\ effect} \\
&\quad \underbrace{-\beta\int_{-\infty}^{+\infty} SED(\tilde{x}_{t+1}|c_{t+1})[\frac{u'''(c_{t+1})}{u'(c_t)}]dc_{t+1}}_{second-degree\ expectation\ dependence\ effect}
\end{aligned} \tag{28}$$

or

$$\begin{aligned}
p_t &= \frac{E_t\tilde{x}_{t+1}}{R^f}-\beta cov_t(\tilde{x}_{t+1},\tilde{c}_{t+1})AR(+\infty)MRS_{+\infty,c_t} \\
&\quad -\beta\int_{-\infty}^{+\infty} SED(\tilde{x}_{t+1}|c_{t+1})AP(c_{t+1})MRS_{c_{t+1},c_t}dc_{t+1},
\end{aligned} \tag{29}$$

where $AP(x) = \frac{u'''(x)}{u'(x)}$ is the index of absolute prudence[7], and

$$\begin{aligned} & E_t\tilde{R}_{t+1} - R^f \qquad (30) \\ = \; & \underbrace{cov_t(\tilde{R}_{t+1}, \tilde{c}_{t+1})[-\frac{u''(+\infty)}{E_t u'(\tilde{c}_{t+1})}]}_{consumption\ covariance\ effect} + \underbrace{\int_{-\infty}^{+\infty} SED(\tilde{R}_{t+1}|c_{t+1}) \frac{u'''(c_{t+1})}{E_t u'(\tilde{c}_{t+1})} dc_{t+1}}_{consumption\ second-degree\ expectation\ dependence\ effect} \end{aligned}$$

or

$$\begin{aligned} & E_t\tilde{R}_{t+1} - R^f \qquad (31) \\ = \; & \beta R^f cov_t(\tilde{R}_{t+1}, \tilde{c}_{t+1}) AR(+\infty) MRS_{\bar{c},c_t} \\ & + \beta R^f \int_{-\infty}^{+\infty} SED(\tilde{R}_{t+1}|c_{t+1}) AP(c_{t+1}) MRS_{c_{t+1},c_t} dc_{t+1}. \end{aligned}$$

Condition (28) includes three terms. The first one is the same as in condition (17). The second term on the right-hand side of (28) is called the "covariance effect." This term involves β, the covariance of asset return and consumption, the Arrow-Pratt risk aversion coefficient and the marginal rates of substitution. The third term on the right-hand side of (28) is called "second-degree expectation dependence effect," which reflects the way in which second-degree expectation dependence of risk affects asset's price through the intensity of downside risk aversion. Again (28) affirms that positive correlation is only a necessary condition for all risk averse and prudent agents to pay a lower price. Equation (30) shows that a positive SED reinforces the positive covariance effect to obtain a positive risk premium.

We state the following propositions without proof. The proofs of these propositions are similar to the proofs of Propositions in Section 3, and are therefore skipped. They are available from the authors.

Proposition 4.1 *Suppose $F(x_{t+1}, c_{t+1})$ and $F(R_{t+1}, c_{t+1})$ are continuous, then the following statements hold:*

(i) $p_t \leq \frac{E_t\tilde{x}_{t+1}}{R^f}$ for any risk averse and prudent representative agent ($u'' \leq 0$ and $u''' \geq 0$) if and only if $(\tilde{x}_{t+1}, \tilde{c}_{t+1}) \in \mathcal{F}_2$;

(ii) $p_t \geq \frac{E_t\tilde{x}_{t+1}}{R^f}$ for any risk averse and prudent representative agent ($u'' \leq 0$ and $u''' \geq 0$) if and only if $(\tilde{x}_{t+1}, \tilde{c}_{t+1}) \in \mathcal{G}_2$;

(iii) $E_t\tilde{R}_{t+1} \geq R^f$ for any risk averse and prudent representative agents ($u'' \leq 0$ and $u''' \geq 0$) if and only if $(\tilde{R}_{t+1}, \tilde{c}_{t+1}) \in \mathcal{F}_2$;

[7]Modica and Scarsini (2005), Crainich and Eeckhoudt (2008) and Denuit and Eeckhoudt (2010) propose $\frac{u'''(x)}{u'(x)}$ instead of $-\frac{u'''(x)}{u''(x)}$ (Kimball, 1990) as an alternative candidate to evaluate the intensity of prudence.

(iv) $E_t\tilde{R}_{t+1} \le R^f$ *for any risk averse and prudent representative agents* ($u'' \le 0$ *and* $u''' \ge 0$) *if and only if* $(\tilde{R}_{t+1}, \tilde{c}_{t+1}) \in \mathcal{G}_2$.

Modica and Scarsini (2005) provide a comparative statics criterion for downside risk in the spirit of Ross (1981).

Definition 4.2 *(Modica and Scarsini 2005) u is more downside risk averse than v if and only if there exists $\lambda > 0$ such that for all x, y*

$$\frac{u'''(x)}{v'''(x)} \ge \lambda \ge \frac{u'(y)}{v'(y)}. \tag{32}$$

More downside risk aversion can guarantee that the decision-maker with a utility function that has more downside risk aversion is willing to pay more to avoid the downside risk increase as defined by Menezes et al. (1980). We can therefore extend Proposition 3.3 as follows:

Proposition 4.3 *Suppose u', v', u''' and v''' are continuous, then the following statements hold:*

(i) $p_t^u \ge p_t^v$ *for all* $(\tilde{x}_{t+1}, \tilde{c}_{t+1}) \in \mathcal{F}_2$ *if and only if v is more Ross and downside risk averse than u;*

(ii) $p_t^u \ge p_t^v$ *for all* $(\tilde{x}_{t+1}, \tilde{c}_{t+1}) \in \mathcal{G}_2$ *if and only if u is more Ross and downside risk averse than v;*

We also obtain the following results for changes in joint distributions.

Proposition 4.4 *(i) Suppose $F(x_{t+1}, c_{t+1})$ is continuous and $E_t^F\tilde{x}_{t+1} = E_t^H\tilde{x}_{t+1}$, then $p_t^F \le p_t^H$ for all risk averse and prudent representative agents if and only if $F(x_{t+1}, c_{t+1})$ is more second-degree expectation dependent than $H(x_{t+1}, c_{t+1})$;*

(ii) Suppose $F(R_{t+1}, c_{t+1})$ is continuous, then for all risk averse and prudent representative agents, $F(R_{t+1}, c_{t+1})$ is more second-degree expectation dependent than $H(R_{t+1}, c_{t+1})$ if and only if the risk premium under $F(R_{t+1}, c_{t+1})$ is greater than under $H(R_{t+1}, c_{t+1})$.

Corollary 4.5 *For all risk averse and prudent representative agents, $E_t^F\tilde{x}_{t+1} \le E_t^H\tilde{x}_{t+1}$ and $F(x_{t+1}, c_{t+1})$ is more second-degree expectation dependent than $H(x_{t+1}, c_{t+1})$ implies $p_t^F \le p_t^H$;*

4.2 C-CAPM for a higher-order representative agent

Ekern (1980) provides the following definition to sign the higher-order risk attitude.

Definition 4.6 *(Ekern 1980) An agent u is N^{th} degree risk averse, if and only if*

$$(-1)^N u^{(N)}(x) \leq 0 \quad for \quad all \quad x, \tag{33}$$

where $u^{(N)}(\cdot)$ denotes the N^{th} derivative of $u(x)$.

Ekern (1980) shows that all agents having utility function with N^{th} degree risk aversion dislike a probability change if and only if it produces an increase in N^{th} degree risk. Risk aversion in the traditional sense of a concave utility function is indicated by $N = 2$. When $N = 3$, we obtain $u''' \geq 0$ which means that marginal utility is convex, or implies prudence. Eeckhoudt and Schlesinger (2006) derive a class of lottery pairs to show that lottery preferences are compatible with Ekern's N^{th} degree risk aversion.

Jindapon and Neilson (2007) generalize Ross' risk aversion to higher-order risk aversion.

Definition 4.7 *(Jindapon and Neilson 2007) u is more N^{th}-degree Ross risk averse than v if and only if there exists $\lambda > 0$ such that for all x, y*

$$\frac{u^{(N)}(x)}{v^{(N)}(x)} \geq \lambda \geq \frac{u'(y)}{v'(y)}. \tag{34}$$

Li (2009) and Denuit and Eeckhoudt (2010) provide context-free explanations for higher-order Ross risk aversion. In Appendix B, we generalize the results of Section 3 and 4 to higher-degree risks and higher order representative agents.

5 Asset prices and two local absolute indexes of risk attitude

If we assume that $\tilde{c}_t$ and $\tilde{c}_{t+1}$ are close enough, then we can use the local coefficient of risk aversion and local downside risk aversion (see Modica and Scarsini, 2005) to obtain the following approximation formulas for (17) and (28):

$$\begin{aligned} p_t &\approx \frac{E_t\tilde{x}_{t+1}}{R^f} + \beta\frac{u''(c_t)}{u'(c_t)}\int_{-\infty}^{+\infty} FED(\tilde{x}_{t+1}|c_{t+1})F_{c_{t+1}}(c_{t+1})dc_{t+1} \\ &= \frac{E_t\tilde{x}_{t+1}}{R^f} - \beta AR(c_t)cov_t(\tilde{x}_{t+1}, c_{t+1}) \end{aligned} \tag{35}$$

and

$$\begin{aligned} p_t &\approx \frac{E_t\tilde{x}_{t+1}}{R^f} + \beta\frac{u''(c_t)}{u'(c_t)}cov_t(\tilde{x}_{t+1}, \tilde{c}_{t+1}) - \beta\frac{u'''(c_t)}{u'(c_t)}\int_{-\infty}^{+\infty} SED(\tilde{x}_{t+1}|c_{t+1})dc_{t+1} \\ &= \frac{E_t\tilde{x}_{t+1}}{R^f} - \beta AR(c_t)cov_t(\tilde{x}_{t+1}, \tilde{c}_{t+1}) - \beta AP(c_t)\int_{-\infty}^{+\infty} SED(\tilde{x}_{t+1}|c_{t+1})dc_{t+1}. \end{aligned} \tag{36}$$

When the variation of consumption is small, (35) implies that absolute risk aversion and covariance determine asset prices while (36) implies that absolute risk aversion, absolute prudence, covariance and SED determine asset prices. We mentioned before that $SED(\tilde{x}|y)$ is related to the 2nd central cross lower partial moment of $\tilde{x}$ and $\tilde{y}$, hence (36) provides a theoretical explanation of the importance of higher-order risk preferences, higher-order moments and comoments in finance. We must emphasize that we obtain only approximations of asset prices when we use Arrow-Pratt measure of risk aversion and Kimball measure of prudence.

6 Concluding remarks on the equity premium puzzle

We discuss the implications of our results on the equity premium puzzle. The major discrepancy between the C-CAPM model predictions and empirical reality is identified as the equity premium puzzle in the literature. As mentioned in Section 3, the key empirical observations of the equity premium puzzle based on (13) can be summarized as follows:

When the representative agent's utility function is the power function, and $\tilde{c}_{t+1}$ and $\tilde{x}_{t+1}$ are conditional lognormally distributed, the observed equity premium can be explained only by assuming a very high coefficient of relative risk aversion. Moreover, it is also difficult to explain the existence of observed high risk premia with the covariance because of the smoothness of consumption over time. However, the equity premium puzzle conclusion is built on specific utility functions and return distributions. Our results show that, for general utility functions and distributions, covariance is not the key element of equity premium prediction. It is very easy to find counter intuitive results. For example, given positively correlated gross return and consumption distributions, a lower Arrow-Pratt coefficient of relative risk aversion may result in a higher equity premium. Alternatively, given a representative agent's preference, a lower covariance between gross return and consumption may result in a higher equity premium. Therefore, (13) is not a robust theoretical prediction of equity premia.

Our results prove that asset pricing's comparative statics imply the following robust predictions:

(a) expectation dependence between payoff and consumption determines asset riskiness rather than covariance;

(b) when a representative agent's risk preference satisfies higher-order risk aversion, more expectation dependence between payoff and consumption is equivalent to a lower price.

(c) when a representative agent's risk preference satisfies higher-order risk aversion, more expectation dependence between gross return and consumption is equivalent to a higher equity premium.

(d) when payoff and consumption are positive expectation dependent, higher risk aversion in the sense of Ross is equivalent to a lower equity price;

Because the comparative Ross risk aversion is fairly restrictive upon preference, some readers may regard (d) as a negative result, because no standard utility functions satisfy such condition on the whole domain. However, some utility functions satisfy comparative Ross risk aversion on some domain. For example, Crainich and Eeckhoudt (2008) and Denuit and Eeckhoudt (2010) assert that $(-1)^{N+1}\frac{u^{(N)}}{u'}$ is an appropriate local index of N^{th} order risk attitude. Nonetheless, some readers may think that because no standard utility functions satisfy these conditions, experimental methods to identify these conditions may need to be developed. Ross (1981), Modica and Scarsini (2005), Li (2009) and Denuit and Eeckhoudt (2010) provide context-free experiments for comparative Ross risk aversion. More research is needed in both directions to develop the theoretical foundations for C-CAPM. This paper takes a first step in that direction. We have proposed a new unified interpretation to C-CAPM, which we have related to the equity premium puzzle problem. Our results are important because C-CAPM shares the positive versus normative tensions that prevail in finance and economics to explain asset prices and equity premia.

7 Appendix A: Proofs of propositions

7.1 Proof of Proposition 3.1

(i): The sufficient conditions are directly obtained from (17) and (18). We prove the necessity using a contradiction. Suppose that $FED(\tilde{x}_{t+1}|c_{t+1}) < 0$ for c^0_{t+1}. Because of the continuity of $FED(\tilde{x}|y)$, we have $FED(\tilde{x}_{t+1}|c^0_{t+1}) < 0$ in interval [a,b]. Choose the following utility function:

$$\bar{u}(x) = \begin{cases} \alpha x - e^{-a} & x < a \\ \alpha x - e^{-x} & a \leq x \leq b \\ \alpha x - e^{-b} & x > b, \end{cases} \tag{37}$$

where $\alpha > 0$. Then

$$\bar{u}'(x) = \begin{cases} \alpha & x < a \\ \alpha + e^{-x} & a \le x \le b \\ \alpha & x > b \end{cases} \tag{38}$$

and

$$\bar{u}''(x) = \begin{cases} 0 & x < a \\ -e^{-x} & a \le x \le b \\ 0 & x > b. \end{cases} \tag{39}$$

Therefore,

$$p_t = \frac{E_t \tilde{x}_{t+1}}{R^f} - \beta \frac{1}{u'(c_t)} \int_a^b FED(\tilde{x}_{t+1}|c_{t+1}) F_{C_{t+1}}(c_{t+1}) e^{-c_{t+1}} dc_{t+1} > \frac{E_t \tilde{x}_{t+1}}{R^f}. \tag{40}$$

This is a contradiction.

(ii) (iii) and (iv): We can prove them using the same approach used in (i).

7.2 Proof of Proposition 3.3

(i): The sufficient conditions are directly obtained from (17), and (21). We prove the necessity by a contradiction. Suppose that there exists some c_{t+1} and c_t such that $\frac{u''(c_{t+1})}{v''(c_{t+1})} > \frac{u'(c_t)}{v'(c_t)}$. Because u', v', u'' and v'' are continuous, there exists a neighborhood $[\gamma_1, \gamma_2]$, such that

$$\frac{u''(c_{t+1})}{v''(c_{t+1})} > \frac{u'(c_t)}{v'(c_t)} \quad for \;\; all \quad (c_{t+1}, c_t) \in [\gamma_1, \gamma_2], \tag{41}$$

hence

$$\frac{-u''(c_{t+1})}{-v''(c_{t+1})} > \frac{u'(c_t)}{v'(c_t)} \quad for \;\; all \quad (c_{t+1}, c_t) \in [\gamma_1, \gamma_2], \tag{42}$$

and

$$-\frac{u''(c_{t+1})}{u'(c_t)} > -\frac{v''(c_{t+1})}{v'(c_t)} \quad for \;\; all \quad (c_{t+1}, c_t) \in [\gamma_1, \gamma_2]. \tag{43}$$

If $F(x, y)$ is a distribution function such that $FED(\tilde{x}_{t+1}|c_{t+1})F_Y(y)$ is strictly positive on interval $[\gamma_1, \gamma_2]$ and is equal to zero on other intervals, then we have

$$p_t^u - p_t^v = \beta \int_{\gamma_1}^{\gamma_2} FED(\tilde{x}_{t+1}|c_{t+1}) F_Y(y) [\frac{u''(c_{t+1})}{u'(c_t)} - \frac{v''(c_{t+1})}{v'(c_t)}] < 0. \tag{44}$$

This is a contradiction.

(ii): We can prove them using the same approach used in (i).

7.3 Proof of Proposition 3.4

(i) The sufficient conditions are directly obtained from (17), and (26). We prove the necessity using a contradiction. Suppose $FED_F(\tilde{x}_{t+1}|c_{t+1})F_{C_{t+1}}(c_{t+1}) < FED_H(\tilde{x}_{t+1}|c_{t+1})H_{C_{t+1}}(c_{t+1})$ for c^0_{t+1}. Owing to the continuity of $FED_F(\tilde{x}_{t+1}|c_{t+1})F_{C_{t+1}}(c_{t+1}) - FED_H(\tilde{x}_{t+1}|c_{t+1})H_{C_{t+1}}(c_{t+1})$, we have $FED_F(\tilde{x}_{t+1}|c^0_{t+1})F_{C_{t+1}}(c^0_{t+1}) < FED_H(\tilde{x}_{t+1}|c^0_{t+1})H_{C_{t+1}}(c^0_{t+1})$ in interval [a,b]. Choose the following utility function:

$$\bar{u}(x) = \begin{cases} \alpha x - e^{-a} & x < a \\ \alpha x - e^{-x} & a \le x \le b \\ \alpha x - e^{-b} & x > b, \end{cases} \tag{45}$$

where $\alpha > 0$. Then

$$\bar{u}'(x) = \begin{cases} \alpha & x < a \\ \alpha + e^{-x} & a \le x \le b \\ \alpha & x > b \end{cases} \tag{46}$$

and

$$\bar{u}''(x) = \begin{cases} 0 & x < a \\ -e^{-x} & a \le x \le b \\ 0 & x > b. \end{cases} \tag{47}$$

Therefore,

$$\begin{aligned} & p^F_t - p^H_t \qquad (48) \\ = \;& \beta \frac{1}{u'(c_t)} \int_a^b [FED_H(\tilde{x}_{t+1}|c_{t+1})F_{C_{t+1}}(c_{t+1}) - FED_F(\tilde{x}_{t+1}|y)F_{C_{t+1}}(c_{t+1})]e^{-c_{t+1}}dc_{t+1} > 0. \end{aligned}$$

(ii): We can prove them using the same approach used in (i).

8 Appendix B: Higher-order risks and higher order representative agents

In this section, we suppose $\tilde{x} \times \tilde{y} \in [a,b] \times [d,e]$, where a, b, d and e are finite. Rewriting $1^{th}ED(\tilde{x}|y) = FED(\tilde{x}|y)$, $2^{th}ED(\tilde{x}|y) = SED(\tilde{x}|y) = \int_d^y FED(\tilde{x}|t)F_Y(t)dt$, repeated integrals yield:

$$N^{th}ED(\tilde{x}|y) = \int_d^y (N-1)^{th}ED(\tilde{x}|t)dt, \; for \; N \ge 3. \tag{49}$$

Definition 8.1 *(Li 2011) If $k^{th}ED(\tilde{x}|e) \geq 0$, for $k = 2, ..., N-1$ and*

$$N^{th}ED(\tilde{x}|y) \geq 0 \ for \ all \ y \in [d, e], \tag{50}$$

then $\tilde{x}$ is positive N^{th}-order expectation dependent on $\tilde{y}$ ($N^{th}ED(\tilde{x}|y)$).

The family of all distributions F satisfying (50) will be denoted by $\mathcal{F}_N$. Similarly, $\tilde{x}$ is negative N^{th}-order expectation dependent on $\tilde{y}$ if (50) holds with the inequality sign reversed, and the totality of negative N^{th}-order expectation dependent distributions will be denoted by $\mathcal{G}_N$. From this definition, we know that $\mathcal{F}_{N-1} \subseteq \mathcal{F}_N$ and $\mathcal{G}_{N-1} \subseteq \mathcal{G}_N$ but the converse is not true. N^{th}-order expectation dependence is related to N^{th}-order central cross lower partial moment of $\tilde{x}$ and $\tilde{y}$ (See, Li (2011) for more details). Several recent researches in finance have focused on estimators of higher-order moments and comoments of the return distribution (i.e. coskewness and cokurtosis) and showed that these estimates generate a better explanation of investors' portfolios. (see, Martellini and Ziemann (2010) for more details).

Definition 8.2 *Define $i^{th}ED_F$ and $i^{th}ED_H$, for $i = 1, .., N$, as the i^{th} expectation dependence under distribution $F(x, y)$ and $H(x, y)$ respectively. Distribution $F(x, y)$ is more first-degree expectation dependent than $H(x, y)$, if and only if $FED_F(\tilde{x}|y)F_Y(y) \geq FED_H(\tilde{x}|y)H_Y(y)$ for all $y \in [d, e]$. Distribution $F(x, y)$ is more N^{th}-order expectation dependent than $H(x, y)$ for $N \geq 2$, if $k^{th}ED_F(\tilde{x}|e) \geq k^{th}ED_H(\tilde{x}|e)$, for $k = 2, ..., N-1$ and*

$$N^{th}ED_F(\tilde{x}|y) \geq N^{th}ED_H(\tilde{x}|y) \ for \ all \ y \in [d, e]. \tag{51}$$

Suppose $(\tilde{x}_{t+1}, \tilde{R}_{t+1}, \tilde{c}_{t+1}) \in [\underline{x}, \overline{x}] \times [\underline{R}, \overline{R}] \times [\underline{c}, \overline{c}]$. We integrate the right-hand term of (16) by parts repeatedly until we obtain:

$$\begin{aligned} cov[u'(\tilde{c}_{t+1}), \tilde{x}_{t+1}] &= \sum_{k=2}^{N} (-1)^k u^{(k)}(\overline{c}) k^{th}ED(\tilde{x}_{t+1}|\overline{c}) \\ &\quad + \int_{\underline{c}}^{\overline{c}} (-1)^{N+1} u^{(N+1)}(c_{t+1}) N^{th}ED(\tilde{x}_{t+1}|c_{t+1}) dc_{t+1}, \ for \ n \geq 2. \end{aligned} \tag{52}$$

Then (11) and (12) can be rewritten as:

$$\begin{aligned} p_t = & \underbrace{\frac{E_t\tilde{x}_{t+1}}{R^f}}_{discounted \ present \ value \ effect} \underbrace{- \beta \sum_{k=2}^{N} k^{th}ED(\tilde{x}_{t+1}|\overline{c})[(-1)^{k+1}\frac{u^{(k)}(\overline{c})}{u'(c_t)}]}_{higher-order \ cross \ moments \ effect} \\ & \underbrace{- \beta \int_{\underline{c}}^{\overline{c}} N^{th}ED(\tilde{x}_{t+1}|c_{t+1})[(-1)^{N+2}\frac{u^{(N+1)}(c_{t+1})}{u'(c_t)}] dc_{t+1}}_{N^{th}-order \ expectation \ dependence \ effect} \end{aligned} \tag{53}$$

$$
\begin{aligned}
&= \frac{E_t\tilde{x}_{t+1}}{R^f} - \beta\sum_{k=2}^{N} k^{th}ED(\tilde{x}_{t+1}|\overline{c})AR^{(k)}(\overline{c})MRS_{\overline{c},c_t} \\
&\quad -\beta\int_{\underline{c}}^{\overline{c}} N^{th}ED(\tilde{x}_{t+1}|c_{t+1})AR^{(k+1)}(c_{t+1})MRS_{c_{t+1},c_t}dc_{t+1}
\end{aligned}
$$

where $AR^{(k)}(x) = (-1)^{k+1}\frac{u^{(k)}(x)}{u'(x)}$ is the absolute index of k^{th} order risk aversion, and

$$
\begin{aligned}
& E_t\tilde{R}_{t+1} - R^f \qquad (54)\\
&= \underbrace{\sum_{k=2}^{N} k^{th}ED(\tilde{R}_{t+1}|\overline{c})[(-1)^{k+1}\frac{u^{(k)}(\overline{c})}{E_t u'(\tilde{c}_{t+1})}]}_{consumption\ cross\ moments\ effect} \\
&\quad + \int_{\underline{c}}^{\overline{c}} \underbrace{N^{th}ED(\tilde{R}_{t+1}|c_{t+1})[(-1)^{N+2}\frac{u^{(N+1)}(c_{t+1})}{E_t u'(\tilde{c}_{t+1})}]}_{N^{th}\ degree\ expectation\ dependence\ effect} dc_{t+1} \\
&= \beta R^f\sum_{k=2}^{N} k^{th}ED(\tilde{R}_{t+1}|\overline{c})AR^{(k)}(\overline{c})MRS_{\overline{c},c_t} \\
&\quad +\beta R^f\int_{\underline{c}}^{\overline{c}} N^{th}ED(\tilde{R}_{t+1}|c_{t+1})AR^{(k+1)}(c_{t+1})MRS_{c_{t+1},c_t}dc_{t+1}.
\end{aligned}
$$

Condition (53) includes three terms. The first one is the same as in condition (17). The second term on the right-hand side of (53) is called the "higher-order cross moments effect." This term involves β, the intensity of higher-order risk aversion, the marginal rates of substitution and the higher-order cross moments of asset return and consumption. The third term on the right-hand side of (53) is called "$N^{th} - degree$ expectation dependence effect," which reflects the way in which N^{th}-degree expectation dependence of risks affects asset price through the intensity of absolute N^{th} risk aversion and the marginal rates of substitution.

We state the following propositions without proof (The proofs of these propositions are similar to the proofs of propositions in Section 3, and are therefore skipped. They are however available from the authors.).

Proposition 8.3 *Suppose $F(x_{t+1}, c_{t+1})$ and $F(R_{t+1}, c_{t+1})$ are continuous, then the following statements hold:*

(i) $p_t \leq \frac{E_t\tilde{x}_{t+1}}{R^f}$ for any i^{th} risk averse representative agent with $i = 2, ..., N+1$ if and only if $(\tilde{x}_{t+1}, \tilde{c}_{t+1}) \in \mathcal{F}_N$;

(ii) $p_t \geq \frac{E_t\tilde{x}_{t+1}}{R^f}$ for any i^{th} risk averse representative agent with $i = 2, ..., N+1$ if and only if $(\tilde{x}_{t+1}, \tilde{c}_{t+1}) \in \mathcal{G}_N$;

(iii) $E_t\tilde{R}_{t+1} \geq R^f$ for any i^{th} risk averse representative agents with $i = 2, ..., N+1$ if and only if $(\tilde{R}_{t+1}, \tilde{c}_{t+1}) \in \mathcal{F}_N$;

(iv) $E_t\tilde{R}_{t+1} \leq R^f$ *for any* i^{th} *risk averse representative agents with* $i = 2, ..., N+1$ *if and only if* $(\tilde{R}_{t+1}, \tilde{c}_{t+1}) \in \mathcal{G}_N$.

Proposition 8.3 suggests that, for an i^{th}-degree risk averse representative agent with $i = 1, .., n = 1$, an asset's price is lowered if and only if its payoff N^{th}-order is positively expectation dependent on consumption. Conversely, an asset's price is raised if and only if it N^{th}-order is negatively expectation dependent on consumption. Therefore, for i^{th}-degree representative agents with $i = 1, .., N+1$, it is the N^{th}-order expectation dependence that determines its riskiness. The next two propositions and Corollary 8.6 have a similar general intuition when compared with those in Section 3.

Proposition 8.4 *Suppose* u', v', u^{N+1} *and* v^{N+1} *are continuous, then the following statements hold:*

(i) $p_t^u \geq p_t^v$ *for all* $(\tilde{x}_{t+1}, \tilde{c}_{t+1}) \in \mathcal{F}_N$ *if and only if* v *is more* i^{th} *risk averse than* u *for* $i = 2, ..., N+1$*;*

(ii) $p_t^u \geq p_t^v$ *for all* $(\tilde{x}_{t+1}, \tilde{c}_{t+1}) \in \mathcal{G}_N$ *if and only if* u *is more* i^{th} *risk averse than* v *for* $i = 2, ..., N+1$*;*

Proposition 8.5 *(i) Suppose* $F(x_{t+1}, c_{t+1})$ *is continuous and* $E_t^F\tilde{x}_{t+1} = E_t^H\tilde{x}_{t+1}$*, then* $p_t^F \leq p_t^H$ *for all* i^{th} *risk averse representative agents with* $i = 2, ..., N+1$ *if and only if* $F(x_{t+1}, c_{t+1})$ *is* N^{th} *more expectation dependent than* $H(x_{t+1}, c_{t+1})$*;*

(ii) Suppose $F(R_{t+1}, c_{t+1})$ *is continuous, then for all* i^{th} *risk averse representative agents with* $i = 2, ..., N+1,$*,* $F(R_{t+1}, c_{t+1})$ *is more* i^{th}*-degree expectation dependent than* $H(R_{t+1}, c_{t+1})$ *if and only if the risk premium under* $F(R_{t+1}, c_{t+1})$ *is greater than* $H(R_{t+1}, c_{t+1})$.

Corollary 8.6 *For all* i^{th} *risk averse representative agents with* $i = 2, ..., N+1$*,* $E_t^F\tilde{x}_{t+1} \leq E_t^H\tilde{x}_{t+1}$ *and* $F(x_{t+1}, c_{t+1})$ *is more* N^{th}*-degree expectation dependent than* $H(x_{t+1}, c_{t+1})$ *implies* $p_t^F \leq p_t^H$*;*

9 References

Andersen S., G. W. Harrison, M. Lau, E. E. Rutstrom, (2011), Intertemporal utility and correlation aversion, working paper, Georgia State University.

Breeden D.T., (1979), An intertemporal asset pricing model with stochastic consumption and investment opportunities, Journal of Financial Economics 7, 265-296.

Campbell J. Y., (2003), Consumption-Based Asset Pricing, Chapter 13 in G. M. Constantinides, M. Harris and R. Stulz eds., Handbook of the Economics of Finance Vol. IB, North-Holland, Amsterdam.

Chabi-Yo F., E. Ghysels, E. Renault (2007), Disentangling the effects of heterogeneous beliefs and preferences on asset prices, Working paper, Ohio State University and North Carolina University.

Chiu, W. H., (2005), Skewness preference, risk aversion, and the precedence of stochastic changes, Management Science 51, 1816-1828.

Cuadras C. M., (2002), On the covariance between functions, Journal of Multivariate Analysis 81, 19-27.

Cochrane, J. H., (2005), Asset Pricing (revised edition). Princeton University Press, Princeton.

Constantinides G. M., D. Duffie ,(1996), Asset pricing with heterogeneous consumers. Journal of Political Economy 104, 219-240

Crainich, D., Eeckhoudt, L., (2008), On the intensity of downside risk aversion, Journal of Risk and Uncertainty 36, 267-276.

Dachraoui K., Dionne G., (2007) Conditions ensuring the decomposition of asset demand for all risk-averse investors, The European Journal of Finance 13, 397-404.

Deck C., Schlesinger H., (2010) Exploring higher order risk effects, Review of Economic Studies 77, 1403-1420.

Denuit, M., Eeckhoudt, L., (2010), Stronger measures of higher-order risk attitudes, Journal of Economic Theory 145, 2027-2036.

Dittmar, R., (2002), Nonlinear pricing kernels, kurtosis preference, and evidence from the cross section of equity returns, Journal of Finance 57, 369-403.

Eeckhoudt, L., Schlesinger H., (2006), Putting risk in its proper place, American Economic Review 96, 280-289.

Eeckhoudt, L., Schlesinger H., (2008), Changes in risk and the demand for saving, Journal of Monetary Economics 55, 1329-1336.

Ekern S., (1980), Increasing Nth degree risk, Economics Letters 6, 329-333.

Embrechts P., (2009), Copulas: A personal view, Journal of Risk and Insurance 76, 639-650.

Galton F. (1886), Regression towards mediocrity in hereditary stature, The Journal of the Anthropological Institute of Great Britain and Ireland 15, 246-263.

Gollier C., (2001), The Economics of Risk and Time. MIT Press, Cambridge.

Gollier C., Schlesinger H., (2002), Changes in risk and asset prices, Journal of Monetary Economics 49, 747-760.

Grossman, S.J., Shiller R.J., (1981), The determinants of the variability of stock market prices, American Economic Review 71, 222-227.

Hansen L. P., Singleton K. J., (1983), Stochastic consumption, risk aversion, and the temporal behavior of asset returns, Journal of Political Economy 91, 249-268.

Harvey, C., Siddique A., (2000), Conditional skewness in asset pricing tests, Journal of Finance 55, 1263-95.

Joe H., (1997), Multivariate Models and Dependence Concepts, Chapman and Hall, London.

Jindapon P., Neilson W., (2007), Higher-order generalizations of Arrow-Pratt and Ross risk aversion: a comparative statics approach, Journal of Economic Theory 136, 719-728.

Kimball M. S., (1990), Precautionary saving in the small and in the large, Econometrica 58, 53-73.

Lehmann E. L., (1966), Some concepts of dependence. The Annal of Mathematical Statistics 37, 1173-1153.

Li, J., (2009). Comparative higher-degree Ross risk aversion, Insurance: Mathematics and Economics, 45, 333-336.

Li, J., (2011), The demand for a risky asset in the presence of a background risk, Journal of Economic Theory, 146, 372-391.

Liu, L., Wang, Z. (2008), A note on an interpretation to consumption-based CAPM, Economics Letters 99(3), 443-445.

Lucas R.E., (1978), Asset prices in an exchange economy, Econometrica 46, 1429-

1445.

Mari D. D., Kotz S., (2001), Correlation and Dependence, Imperical College Press, London.

Martellini L., Ziemann V., (2010), Improved estimates of higher-order comoments and implications for portfolio selection, Review of Financial Studies 23, 1467-1502.

Menezes C., Geiss C., Tressler J., (1980), Increasing downside risk, American Economic Review 70, 921-32.

Mitton T., Vorkink, K., (2007), Equilibrium underdiversification and the preference for skewness, Review of Financial Studies 20, 1255-1288.

Modica S., Scarsini M., (2005), A note on comparative downside risk aversion, Journal of Economic Theory 122, 267-271.

Pellerey F, Semeraro P., (2005) A note on the portfolio selection problem, Theory and Decision 59: 295-306.

Ross S. A., (1981), Some stronger measures of risk aversion in the small and in the large with applications, Econometrica 49, 621-663.

Rubinstein, M., (1976), The valuation of uncertain income streams and the pricing of options, Bell Journal of Economics 7, 407-425.

Sibuya M., (1960), Bivariate extreme statistics, I, Annals of the Institute of Statistical Mathematics 11, 195-210.

Tesfatsion L., (1976), Stochastic dominance and the maximization of expected utility, The Review of Economic Studies 43, 301-315.

Wright R., (1987), Expectation dependence of random variables, with an application in portfolio theory, Theory and Decision 22, 111-124.