Empirical Evaluation of Investor Rationality
in the Asset Allocation Puzzle

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Abstract

We examine the portfolio-choice puzzle posed by Canner, Mankiw, and Weil (1997). The idea is to test a conclusion reached by Elton and Gruber (2000), stating that a bonds/stocks ratio which decreases in relation to risk tolerance does not necessarily mean a contradiction of modern portfolio-choice theory and does not cast doubt on the rationality of investors. From data on the portfolio composition of 470 clients of a Canadian brokerage firm, we obtain that the bonds/stocks ratio does decrease in relation to risk tolerance. We also verify the existence of the two-fund separation theorem in the assets data available to the investors in our sample.

Keywords: Investor rationality, asset allocation puzzle, risk tolerance, separation theorem, bonds/stocks ratio.

JEL classification: C13, D12, D80, G11, G23.

Résumé

Nous analysons l'énigme du choix de portefeuille proposée par Canner, Mankiw et Weil (1997). L'idée est de tester une conclusion de Elton et Gruber (2000) stipulant qu'un ratio obligations/actions décroissant en fonction de la tolérance face au risque n'implique pas nécessairement une contradiction par rapport à la théorie moderne de choix de portefeuille et n'introduit pas de doute sur la rationalité des choix individuels. À partir de données de 470 portefeuilles individuels d'une entreprise de courtage canadienne, nous obtenons que le ratio obligations/actions décroît en relation avec la tolérance face au risque. Nous vérifions aussi l'existence du théorème de séparation à deux fonds dans les données sur les actifs disponibles aux investisseurs de notre échantillon.

Mots clés : Rationalité de l'investisseur, énigme du choix de portefeuille, tolérance au risque, théorème de séparation, ratio obligations/actions.

Classification JEL: C13, D12, D80, G11, G23.

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Empirical Evaluation of Investor Rationality
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I Introduction

This research proposes an empirical solution to the asset-allocation puzzle posed by Canner, Mankiw, and Weil (1997). These authors conclude that the recommendations of some financial advisors are inconsistent with rational allocation as advocated by the modern portfolio theory (MPT). They claim that if, in the presence of a risk-free asset, the bonds/stocks ratio was seen to decrease in relation to risk tolerance (measured by the proportion invested in stocks), this would contradict the conclusion of the two-fund separation theorem which predicts a constant bonds/stocks ratio at all levels of risk tolerance.

Several previous studies have attempted to solve this asset-allocation puzzle by adopting three main lines of research. The first relies on dynamic asset-allocation models: the individual investor tries to maximize his expected utility, while keeping an eye on evolving future returns on the different financial assets (bonds, stocks, and cash). On this topic, we find the works of Bajeux-Besnainou, Jordan and Portait (2001, 2003), of Brennan and Xia (2000, 2002), of Campbell and Viceira (2001), and of Wachter (2003). These studies look at different explanations such as particular specifications of utility function (CRRA, HARA); the link between different financial assets; the inflation factor or the investor's time horizon (finite number of years or infinite horizon).

The second main line of research groups single period theoretical studies. Among these studies, we may cite the contribution of Boyle and Gurthie (2005) who come to the conclusion that the correlation between the return on stocks and human capital could generate a decreasing bonds/stocks ratio, even in a context which authorizes short selling and offers a risk-free asset. We may also cite Elton and Gruber (2000) who have shown that disallowing short-selling and/or eliminating the risk-free asset can explain the bonds/stocks ratio's negative slope with regard to risk tolerance.

Finally, the third main line of research contains empirical studies like the one by Siebenmorgen and Weber (2000) who turn their attention to the asset allocations advocated by German financial advisors. These authors conclude that the choices made by these advisors are rational when viewed through the lens of a behavioural finance model like the one presented by Benartzi and Thaler (2001). We should also cite Shalit and Yitzhaki (2003) who used the same data as Canner et al. (1997) to test investor rationality. Using second-order stochastic dominance as the portfolio optimization criterion, they conclude that the recommendations made by financial advisors were rational.

Our study fits in with the last two lines of research and, more particularly, with the contribution of Elton and Gruber (2000). Our main difference is to use individuals' portfolio choices instead of recommendations from financial advisors. In the second section, we analyze individual investor rationality in a mean-variance single-period framework. We then display the results obtained from an original database of 470 Canadian investor portfolios regarding the relation

between bonds/stocks ratios and risk tolerance. Finally, we test for the presence of the separation theorem in order to reach our conclusion on investor rationality.

II Rationality of economic agents

Showing that a bonds/stocks curve which declines in relation to risk tolerance could be consistent with modern portfolio theory, Elton and Gruber (2000) came to the conclusion that the allocations suggested by the financial advisors[1] in Canner et al. (1977) may be rational. This divergence from the conclusions of Canner et al. (1997) is essentially a function of the context considered: whether a risk free asset is present or not and whether short selling is allowed or not.

The possibility of selling short or not can be cited as one of the rules governing the market. Switching from a context which does authorize short selling to one which does not entails a host of changes. The first consequence is a reduction of the investor's range of possible combinations. The second is related to determining the optimal mean-variance combinations for this same investor. In this respect, it is worth noting that restricting short sales makes the optimal-allocation problem harder to solve. The analytical solution found in a context where short-selling is allowed ceases to be valid when negative proportions of the financial assets are disallowed. In such a case, one alternative means of solving optimal-allocation problems would be through numerical methods.

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[1] The reference is to Fidelity, Jan Bryant Quinn and Merrill Lynch and to recommendations in the New York Times.

Obviously, permission to short-sell would be preferable for the reasons cited above. However, current financial market practices should also be kept in view. It is in fact rare to find markets that allow unlimited short-selling by individuals. The cost of short-selling is usually higher. Besides, this practice is not encouraged in many brokerage firms—mainly owing to the extra costs and higher risks short positions entail. Such risks are higher in illiquid markets. Finally, we should point out that Jones and Lamont (2002) and Lamont (2004) have confirmed the existence of regulations banning short sales on some financial markets. This confirmation is based on the observation of several excessively overvalued stocks on these markets.[2] However, it is not obvious that these regulations act to restrict all markets, especially the market of individual investors covered by our study.

We now shift our attention to the hypothesis concerning the existence of a risk-free asset. When the short-selling is allowed, whether or not a risk-free asset exists will not be an important factor in solving the optimal-allocation problem. It is in fact possible to determine analytically the optimal portfolios for all risk levels. But from a more practical point of view, assumptions concerning the existence of such an asset on financial markets will be less obvious. Though a huge number of corporate and government bonds with nominally constant interest rates do exist on the market, it would be foolhardy to affirm the existence of a totally risk-free asset in the economy. Fluctuating inflation rates cause the yield (in real terms) offered by these bonds to vary over time. However, focusing on a short time horizon might be synonymous with a weak

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[2] Jarrow (1980) has shown that regulations banning short-selling imply a price hike in risky assets when all individual investors consider the same variance-covariance matrix.

variation in the inflation rate and, consequently, could favour the hypothesis that a risk-free asset does exist.

We need to examine what effect each of these contexts will have on the optimal allocation of assets or, more precisely, on how the bonds/stocks ratio will vary in relation to risk tolerance. Figure 1 sums up the cases analyzed by Elton and Gruber (2000). With a risk-free asset and the possibility of short-selling, we should expect a constant bonds/stocks ratio for all investors (no matter what their level of risk tolerance). This ratio becomes a monotone function (either increasing or decreasing) when considering real-term returns (synonymous with the absence of a risk-free asset in an inflationary economy). Restrictions on short-selling will have two possible effects: either a decreasing bonds/stocks ratio in function of risk tolerance or a bonds/stocks ratio which will first increase for relatively low levels of risk tolerance and then later decrease.

Thus, as Elton and Gruber (2000) point out, when the slope of the bonds/stocks ratio is observed to be negative in relation to risk tolerance this should not be understood as a non-optimal investor choice. On the contrary, the theoretical contexts leading to this observation are apparently more in line with practice.

(Figure 1 here)

III Empirical relation between investors' choices and their risk tolerance

A Data

In our attempt to solve the asset allocation puzzle posed by Canner et al. (1997), we used data obtained from a Canadian brokerage firm specializing in financial services to individual investors. The originality of this database is that it contains positions chosen by individual investors rather than products offered by brokers as in Canner et al. (1997). These data contain the portfolio composition[3] of 470 of that firm's clients in July 2000, along with their individual characteristics such as age, investment knowledge, income, and investment objectives.

Table 1 presents these data.

(Table 1 here)

It appears that 58% of clients claim to have "acceptable" investment knowledge. The percentage of those rating their knowledge as "good" stands at 34%, whereas those claiming "excellent knowledge" represent 4% of the sample. The remaining 4% have no knowledge on the subject. Another aspect which drew our attention concerns the types of accounts held by these individual investors. This datum could, in effect, give us a better idea of each investor's risk aversion. From our observations, we find that all the clients hold a checking account.[4] Of these clients 67% also

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[3] The portfolio is divided into three classes of assets: Treasury bills, bonds, and stocks.

[4] We should emphasize that these categories are not mutually exclusive. This will be important in the statistical analysis.

have a pension fund account; 15% hold a margin account (short-selling); and 2% have both a margin and a pension fund account. Distribution of investors' total net assets is given in table 2.

(Table 2 here)

One last datum likely to influence asset allocation involves the financial advisor with whom each of the investors deals. In our case, the 470 investors selected use the services of 4 financial advisors. Table 3 shows the proportion of clients served by each advisor and provides a brief description of the advisor. Note that the clients of advisors 3 and 4 have been pooled, because these two advisors work together, have the same management style, and the same type of clientele (age, wealth…).

(Table 3 here)

B Risk tolerance and portfolio-choice

Drawing on our data, it is easy to construct Figure 2 showing the bonds/stocks ratios held by 358 of the 470 clients[5] in terms of their risk tolerance, as measured by the proportion of assets invested in stocks.

(Figure 2 here)

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[5] Of course, we cannot use bonds/stocks ratios for those clients who hold no stocks in their portfolio.

Figure 2 shows a negative slope for the bonds/stocks ratio in relation to the proportion of the portfolio invested in stocks. This observation should not, however, imply a confirmation of the paradox mentioned by Canner et al. (1997). Indeed, as a preliminary step in our study, we must check whether the proportion of assets invested in stocks is a good measure of risk tolerance. A second index of risk tolerance "*T(Ind)*" is then calculated for each of the clients, based on their investment objectives:

(1) $$T(Ind) = \frac{1*inc\% + 2*growth\% + 3*spec\%}{3} \quad \text{where} \quad \frac{1}{3} \leq T(Ind) \leq 1.$$

where:

T(Ind): indirect measurement of risk tolerance;

inc%: investment objective in income securities (percentage of total portfolio);

growth%: investment objective in growth securities (percentage of total portfolio);

spec%: investment objective in speculative securities (percentage of total portfolio);

with *inc% + growth% + spec % = 100%.*

Notice that the average of this second risk-tolerance index is 56% for all the individual investors considered, as compared to an average of 57% for the direct measurement of risk tolerance.

With this indirect measurement of risk tolerance, we can perform the following regressions to test the equivalence between the two measurements:

(R1) $$Y = \beta_0 + \beta_1 T(Dir) + \beta_2 Z + \varepsilon_1$$

(R2) $$Y = \beta_3 + \beta_4 T(Ind) + \beta_5 Z + \varepsilon_2$$

where:

Y: proportion invested in bonds;[6]

T(Dir): direct measurement of risk tolerance, measured by the proportion invested in stocks;

T(Ind): indirect measurement of risk tolerance;

Z: vector of the individual characteristics of each investor: age, income, size of portfolio,

investment knowledge…

The results of these two regressions are presented in Table 4 (R1 and R2).

(Table 4 here)

The equivalence test for the two tolerance measurements (comparison between the parameters $\hat{\beta}_1$ and $\hat{\beta}_4$) indicates that they are not statistically different at a 95% confidence level. This observation is later reaffirmed by ensuring that the results obtained are not due to an econometric specification problem.[7] Thus, the proportion invested in stocks serves as a good measurement of risk tolerance and cannot be advanced as a plausible explanation of the paradox posed by Canner et al. (1997).

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[6] The proportion invested in bonds is used as a dependent variable in order to include the maximum observations, i.e. 405 clients with all the information needed in the regression. In effect, using the bonds/stocks ratio as the dependent variable would reduce the number of observations by 23% (93 of the 405 clients) because these clients do not hold stocks. When we estimated the model with 312 observations, the results obtained were the same whether based on the proportion invested in bonds or the bonds/stocks ratio. Neither coefficient differs significantly from those presented in Table 4. (Details are available upon request.)

[7] For example, when we add $E\big(T\big(Dir\big)\big)$ in (R1), the coefficient of $T\big(Dir\big)$ becomes -1.05 with a *statistic t =* -96.125. Other results are available upon request.

On the basis of this observation, we propose to analyze the variation in the bonds/stocks ratio in relation to the risk tolerance of each of the 405 clients. Our goal is to explain individuals' asset allocation in terms of their respective risk tolerance and certain other personal variables (age, annual income…). From regression R1, it is easy to check whether the bonds/stocks ratio remains constant for all the clients considered. This ratio (designated r) can be expressed as follows:

(2)
$$r = \frac{Y}{T(Dir)} = \frac{\beta_0 + \beta_1 T(Dir) + \beta_2 Z}{T(Dir)} = \beta_1 + \frac{\beta_0 + \beta_2 Z}{T(Dir)}.$$

Thus, the variation of ratio r relative to risk tolerance is equal to:

(3)
$$\frac{dr}{dT(Dir)} = -\frac{\beta_0 + \beta_2 Z}{[T(Dir)]^2}.$$

Testing whether ratio r is constant for all individual investors comes down to testing whether $\hat{\beta}_0$ and $\hat{\beta}_2$ are statistically and jointly equal to zero.[8] The Fisher test rejects this hypothesis at a confidence level of 95% for both regressions R_1 and R_2 and shows a bonds/stocks ratio which declines in relation to the risk tolerance of the individuals considered (see Table 4). In Section V, we shall introduce a third measure of risk tolerance to test the robustness of our results.

As Elton and Gruber (2000) note, this rejection of the hypothesis assuming a constant bonds/stocks ratio for all individual investors, based on modern portfolio-choice theory, is not sufficient to conclude for the presence of an asset allocation puzzle. It would be advisable to test

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[8] A Hausman test was performed in order to screen the regression for endogeneity problems.

also whether the investors considered made their portfolio choices in a context supporting the two-fund separation theorem—thus weighing the rationality of their behaviour.

IV Test of the separation theorem

One of the basic hypotheses used in this research consists in accepting the mean-variance model which allows two-fund separation for any increasing and concave utility function, when return distributions belong to the elliptical family.[9] Checking that returns of stocks, bonds, and cash belong to the elliptical family of distributions and testing for the separation theorem will allow us to consolidate our conclusions concerning the rationality of portfolio holders. It is advisable to first present the assets data that the individual investors in our study may have used in their portfolio selection.

A Returns on financial assets

To evaluate the returns that individuals in our database considered when making their portfolio selections, we turned to the performance records of three Canadian mutual funds available to the investors of this study. Returns achieved by mutual funds do, in fact, serve as a good indicator for the different financial markets (Barras, Scaillet, and Wermers, 2005).[10]

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[9] See Owen and Rabinovitch (1983).

[10] Barras et al. (2005), in their survey, based uniquely on data from the U.S., find that just 20% of all equity mutual funds obtain a negative performance.

In evaluating the returns considered by the 470 investors in our sample, we first look at the returns obtained by *Ferique Equity, Ferique Bonds,* and *Ferique Short Term Income.*[11] The information available in each of these funds's prospectus will give a better understanding of our selection. The objective set by *Ferique Equity* is to obtain long-term capital gains by investing in the stocks of Canadian companies. *Ferique Bonds*, for its part, aims to provide a steady stream of high income and, occasionally, some capital gain from investments in Canadian bonds. Its portfolio is composed of Canadian bonds issued by the Canadian government, provinces, municipalities, and corporations. Finally, *Ferique Short Term Income* proposes to provide current income, while protecting capital and maintaining high liquidity. Its portfolio is composed, up to 80%, of Canadian debt securities maturing in under 6 months. Statistics on these returns are presented in Table 5.

(Table 5 here)

Returns from the three funds were observed monthly between January 1995 and June 2000. Three remarks justify our selection. First, remember that, for each investor, the portfolio composition considered was that from the month of July 2000. It is thus reasonable to consider returns preceding that date. A second question about these data concerns the frequency with which they were observed. On this point, note that several empirical works[12] on the problem of portfolio selection make use of monthly returns. Finally, an observation period of about 5 years

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[11] These returns are available on the *Ferique* funds site (www.ferique.com).

[12] According to Elton and Gruber (2000), *"in finance, it is common to use monthly intervals to measure returns used in estimating expected returns, variances and covariances."*

would seem to be a judicious choice. A shorter period will produce less accurate results, whereas estimations based on a very long period run the risk of being affected by changes of regime.

Also worth noting is the strong correlation between returns from the three funds considered and those obtained by certain standard indices over the same period: from January 1995 to June 2000. Indeed, a 91.97% correlation coefficient is observed between the returns generated by *Ferique Equity* and those on the Toronto Stock Exchange index (TSE300). We obtain a 87.24% correlation coefficient between returns on *Ferique Bonds* and those reported by the Scotia Capital (Overall Universe) index.[13] Finally, the correlation between returns from *Ferique Short Term Income* and the average return on one-month Treasury bills[14] stands at 80.39%.

Once these returns have been defined historically, we shall then be in a position to see whether they can be considered part of a family of elliptical distributions.

B Ellipticality test for returns on financial assets

Two families of tests are generally used to determine the nature of multivariate distributions: the Jarque-Bera (1987) type and the Mardia (1970) type. For an application of the first type, we refer to Kilian and Demiroglu (2000). After first calculating the degree of skewness and kurtosis for each separate random variable, an aggregation of these univariate results produces statistics related to the multivariate distribution. Tests of the Mardia type allow a direct calculation of the

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[13] Returns related to the TSE300 Index and the Scotia Capital Index were drawn from the Datastream base.

[14] This datum was obtained from the Bank of Canada site: identifier V122529 in the CANSIM directory.

multivariate skewness as well as the multivariate kurtosis. These tests have the advantage of taking into account the correlation between the different random variables in the joint distribution. These tests also lead to the same results as those obtained with the Jarque-Bera type test in the univariate case and seem more suitable to the multivariate case.

A Mardia-type test is based on two statistics: the multivariate skewness (*MSK*) and the multivariate kurtosis (*MKU*). First, we present these two statistics. We then describe the methodology adopted to determine the nature of the joint distribution of the returns on stocks, on bonds, and on cash.

Let *N* risky assets be observed over *T* periods; we have *T* vectors $R_1, R_2, \ldots, R_T$; each vector contains the returns observed at a given date for the *N* risky assets considered. We can note by d_{ts} all elements of the matrix $\left(R_t - \overline{R}\right)' S^{-1} \left(R_s - \overline{R}\right)$ for all *t* and *s* contained between 1 and *T* where $\overline{R}$ is the vector of average returns and *S* the variance-covariance matrix of the same returns. The multivariate skewness and kurtosis are calculated as follows:

(4)
$$MSK = \frac{1}{T^2} \sum_{t=1}^{T} \sum_{s=1}^{T} d_{ts} \quad \text{and} \quad MKU = \frac{1}{T} \sum_{t=1}^{T} d_{tt}^2 .$$

These two statistics will serve as the basis for determining the distribution of the returns observed for stocks, cash and bonds. The first step consists in calculating statistics *MSK* and *MKU* (noted respectively as *MSK_obs* and *MKU_obs*) relative to the series of returns observed. The second step is based on simulations. We make *T* drawings of *N* random variables according to a precisely determined distribution (multivariate Student, multivariate normal, …), taking into account a

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variance-covariance matrix equivalent to the one linked to the observations. Based on these simulated data, it is possible to calculate statistics *MSK* and *MKU*. The results obtained will be noted as $MSK_{sim,1}$ and $MKU_{sim,1}$. Repeating these simulations *M* times will allow us to obtain the following vectors: $MSK_{sim} = [MSK_{sim,1}, MSK_{sim,2}, ..., MSK_{sim,M}]$' and $MKU_{sim} = [MKU_{sim,1}, MKU_{sim,2}, ..., MKU_{sim,M}]$'. We next classify the elements of these two vectors to find the vectors MSK_{sim}^{ord} and MKU_{sim}^{ord}. Finally, noting the fact that MSK_{obs} is bounded by the $\left[\dfrac{\alpha}{2}M+1\right]^{th}$ value

and the $\left[\left(1-\dfrac{\alpha}{2}\right)M-1\right]^{th}$ value of vector MSK_{sim}^{ord} and that MKU_{obs} is bounded by the

$\left[\dfrac{\alpha}{2}M+1\right]^{th}$ value and the $\left[\left(1-\dfrac{\alpha}{2}\right)M-1\right]^{th}$ value of vector MKU_{sim}^{ord}, this allows us to conclude

that the returns observed follow the multivariate distribution simulated at a confidence level of (1 − α).

We shall now apply the methodology described above to our data: monthly returns noted between January 1995 and June 2000 for *Ferique Equity, Ferique Bonds,* and *Ferique Short Term Income.* The statistics related to the returns observed stand at 2.2493 and 15.7729 respectively for the multivariate skewness and kurtosis. These values lead us to conclude, at a 95% confidence level, that the returns observed do not reject the multivariate Student distribution with 10 degrees of freedom.[15] In fact, the confidence interval related to the multivariate skewness is equal to [0.3915; 4.9090], whereas that related to the multivariate kurtosis corresponds to [3.4822; 23.9927].

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[15] These results were obtained based on 9,999 simulations.

Finding that the returns on stocks, bonds, and cash may correspond to one of the elliptical distributions (Student distribution with 10 degrees of freedom) allows us to test the Black-CAPM by assuming there is no risk free asset and no restriction on short selling—both reasonable assumptions for the observed investment environment of our initial data set.

C Black-CAPM test with non-gaussian returns

Beaulieu, Dufour, and Khalaf (2003) have presented a test of the Black's Capital Asset Pricing Model (BCAPM) with possibly non-gaussian returns. It seems advisable to adopt their methodology, since we are dealing with three risky assets whose returns seem to correspond to a Student distribution. We shall now present the methodology used to test the Black-CAPM (BCAPM). A brief introduction to the model is required to explain the notations to be used.

Note as R_{it}, $i = 1, \ldots, n$, the returns on n risky assets during period t (stretching from 1 to T) and as $\tilde{R}_{Mt}$ the returns on the market portfolio. The BCAPM test will thus be based on the following model:

(5) $$R_{it} = a_i + b_i \tilde{R}_{Mt} + u_{it}; t = 1, \ldots, T, i = 1, \ldots, n$$

where u_{it} designates the error term. In fact, testing BCAPM comes down to checking whether there is a scalar γ (return on the zero-beta portfolio whose composition is unknown to us) such that:

(6) $$H_{BCAPM}: a_i = \gamma(1 - b_i), \ \forall i = 1, \ldots, n.$$

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Model (5) can be re-written under the matricial form:

(7)
$$Y = XB + U$$

with:

$$Y = [R_1, \ldots, R_n], \ X = [\iota_T, \tilde{R}_M];$$

$$R_i = (R_{1i}, \ldots, R_{Ti})' \ ; \ \tilde{R}_M = (\tilde{R}_{1M}, \ldots, \tilde{R}_{TM})' \ \text{and} \ \iota_T = (1, \ldots, 1)'.$$

Finally, the hypothesis test presented in (6) is based on the calculation of the quasi likelihood ratio calculated as follows:

(8)
$$LR_{BCAPM} = T \ln(\Lambda_{BCAPM}) \text{ with } \Lambda_{BCAPM} = \left|\hat{\Sigma}_{BCAPM}\right| / \left|\hat{\Sigma}\right|,$$

where:

$$\hat{\Sigma} = \hat{U}'\hat{U}/T \ ; \ \hat{U} = Y - X\hat{B} \text{ and } \hat{B} = (X'X)^{-1} X'Y,$$

and $\hat{\Sigma}_{BCAPM}$ designates the $\hat{\Sigma}$ estimator in the constrained model which verifies hypothesis (6).

One of the basic hypotheses of the methodology of Beaulieu et al. (2003) is the possibility of re-writing vector $U_t = (u_{1t}, \ldots, u_{nt})'$ as the product of an unknown triangular matrix J and a vector $W_t = (W_{1t}, \ldots, W_{nt})'$ whose joint distribution is fully specified. We thus obtain the following equalities:

(9)
$$U_t = JW_t$$

(10)
$$\Sigma = JJ',$$

where Σ designates the variance-covariance matrix of vector U_t.

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Given this hypothesis advanced by Beaulieu et al. (2003), the likelihood ratio, defined by expression (8), is distributed as follows:

(11) $$LR(\gamma_0) = T \ln\left(\left|W'M_0W\right|/\left|W'MW\right|\right)$$

where:

$$M = I - X(X'X)^{-1}X'$$

and

$$M_0 = M + X(X'X)^{-1}H'\left[H(X'X)^{-1}H'\right]^{-1}H(X'X)^{-1}X'$$

$$\text{with } H = \begin{bmatrix} 1 & \gamma_0 \end{bmatrix} \text{ and } W = \begin{bmatrix} W_1,\ldots,W_T \end{bmatrix}'.$$

The BCAPM test thus comes down to first setting a value for the scalar γ_0 and then calculating the likelihood ratio it entails. The second step consists in simulating N drawings for the multivariate distribution W. For each of these drawings, we calculate the likelihood ratio as defined by expression (11). Calculation of the specific *p-value* of the scalar γ_0 is obtained as follows:

(12) $$\hat{p}_N\left(LR(\gamma_0)\mid v\right) = \frac{\hat{G}_N(\gamma_0,v)+1}{N+1},$$

where v designates the parameters of the distribution used during the simulations (such as the degree of freedom during simulation of a Student distribution) and $\hat{G}_N(\gamma_0,v)$ corresponds to the number of ratios resulting from the simulations which exceed the ratio calculated based on the observations.

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This calculation of *p-values* is repeated for several possible values of γ_0 and the *p-value* of the

BCAPM test is obtained as follows:

(13)
$$\hat{p}_N^*\left(LR_{BCAPM}\,|\nu\right) = \sup_{\gamma_0} \hat{p}_N\left(LR(\gamma_0)\,|\nu\right).$$

In the end, the decision rule concerning the hypothesis test cited in (6) consists in comparing the

p-value transferred to (13) and the level of significance α considered: if the *p-value* exceeds α,

the BCAPM hypothesis is not rejected.

We now apply this BCAPM test to our particular context. Considering *Ferique Balanced* as a

market portfolio[16] based on the results related to the distribution of returns from *Ferique Equity*,

Ferique Bonds, and *Ferique Short-Term Income* (Student distribution at 10 degrees of freedom),

we reach the conclusion that the BCAPM is not rejected at a 99% level of confidence when 9,999

simulations of the multivariate Student distribution at 10 degrees of freedom are considered (see

Table 6 for a detailed presentation of the empirical results).

(Table 6 here)

To consolidate our conclusions, certain robustness tests are advisable. Indeed, our previously

results might depend on approximations of risk tolerance and returns from stocks, bonds, cash

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[16] The monthly returns between January 1995 and June 2000 are considered. The portfolio of this fund is composed

of stocks, bonds and short-term assets. Statistics of these returns are available in Table 5.

and the market portfolio. These results could also arise from the methodology used to test the

BCAPM.

V Robustness of results

Our first robustness test concerns the measurement of risk tolerance. The results obtained by the

indirect measurement of risk tolerance defined in (1) might in fact depend on the coefficients

assigned. Erroneous interpretation of these coefficients may occur: it may be arbitrary to suppose

that an investor placing his money in speculative assets is 3 times more risk tolerant than the one

who places his money in income assets.

We thus propose a third risk-tolerance measurement which is defined as follows:

$$(14) \qquad T(Obs) = \frac{rmon * mon\% + rbond * bond\% + rstc * stc\%}{rstc}$$

where *mon%, bond%* and *stc%* represent respectively the proportions each investor holds in

money, bonds, and stocks. *rmon, rbond* and *rstc* designate the average returns on the money,

bonds, and stocks considered by all the investors.[17]

A more risk-tolerant investor will tend to place his wealth in high-risk financial assets, which are

synonymous with higher returns. Thus an investor's observed portfolio returns should be

indicative of his risk tolerance. The regression of the proportion invested in bonds in relation to

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[17] In our case, these average monthly returns amount to 0.4%, 0.7%, and 1.34% for money, bonds and stocks

respectively, supposing that individual investors turn to *Ferique* funds in making their portfolio selections.

this new risk-tolerance measurement is presented in Table 4 (R3). Defined by average returns on the different financial assets and by the portfolio's composition, this composite measurement also indicates a negative slope of the bonds/stocks ratio relative to risk tolerance, with a 95% confidence level.

Our second robustness test consists in using other approximations to calculate returns from stocks, bonds, liquidities, and the market portfolio. As an alternative to *Ferique* funds, we use the returns generated by *Talvest* funds or by *TD* funds between January 1995 and June 2000.[18]

To evaluate returns on stocks, we selected *Canadian Equity Value* from the *Talvest* funds. As indicated in its prospectus, the fund's objective is to obtain higher than average long-term capital growth, by investing mainly in Canadian equity securities. The bond yield is evaluated by the performance of the *Talvest Bond Fund.* This fund's objective is to maintain capital while obtaining high current income, by investing mainly in bonds, debentures, notes, and other debt instruments of financial institutions, corporations, and Canadian governments. Approximation of the return on cash is based on returns generated by the *Talvest Money Market Fund.* This fund proposes to obtain high income, while protecting both capital and liquidity, by investing mainly in high-quality, short-term debt securities issued or guaranteed by the government of Canada or by one of its provinces. Descriptive statistics of the returns of these funds are presented in Table 5.

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[18] Direct access to the performance of the funds selected are available on the two following Web sites: www.talvest.com and www.tdcanadatrust.com.

We should notice the high correlations between *Talvest* funds and other funds having the same objectives. For example, we obtain a 84.14% correlation coefficient between returns on *Ferique Equity* and *Canadian Equity Value* from *Talvest* funds. Other correlations between returns on the funds considered are available in Table 7.

(Table 7 here)

The first test applied to the returns on these three funds does not permit us to reject the null hypothesis of elliptically distributed returns at the 95% confidence level. The multivariate skewness in the joint distribution of these returns actually amounts to 1.7369, whereas the multivariate kurtosis is equal to 17.9584, These statistics range within the intervals at the 95% confidence level for the multivariate Student distribution with 15 degrees of freedom[19], based on 9,999 simulations.

This non-rejection of the ellipticity of the returns allows us to test for the Black's Capital Asset Pricing Model (BCAPM), applying the methodology used in Section IV. The market portfolio considered in this test corresponds to *Talvest's Canadian Asset Allocation Fund* whose stated objective is to obtain long-term stable capital growth, by investing mainly in a balanced portfolio composed of Canadian equity and debt securities, including money market instruments. The BCAPM test on *Talvest* funds, and on 9,999 simulations of the multivariate Student distribution,

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[19] The confidence interval for the multivariate skewness is [0.3327; 3.3099], whereas that for the multivariate kurtosis is [13.0665; 20.8151] at the 95% confidence level and for the multivariate Student distribution at 15 degrees of freedom.

does not reject the null hypothesis of the presence of the separation theorem at the 99% confidence level (details are in Table 6).

The methodology cited above was also applied to *TD* funds. *TD Canadian Money Market Fund, TD Canadian Equity Fund*, and *TD Canadian Bond Fund* were selected to evaluate, respectively, the return on cash, stocks, and bonds traded in Canada.

The evaluation based on these data also prevents us from rejecting the elliptical distribution of returns at the 95% confidence level. Indeed, we obtain a skewness equal to 2.0406 and a kurtosis of 17.1734 for the joint distribution, whereas the intervals of confidence obtained for a multivariate Student distribution with 8 degrees of freedom are [0.4466; 6.6795] and [13.8692; 27.0550] respectively.

This observation leads us to test the BCAPM based on the *TD Balanced Fund* as a market portfolio. Applying the method cited above and based on our 9,999 simulations, we do attain the non-rejection of the BCAPM at the 99% confidence level.

It would also be advisable to apply a second methodology for testing the BCAPM in the presence of non-Gaussian returns. This second technique, drawn from the work of Zhou (1993), differs from that of Beaulieu et al. (2003) by its non-separation between the nuisance terms (the unknown triangular matrix *J*) and the *W* vector whose joint distribution is fully specified. In adopting this second methodology, the distribution of the likelihood ratio will be:

(15) $$LR(\gamma_0) = T \ln\left(\left|\hat{U}'M_0\hat{U}\right| \Big/ \left|\hat{U}'M\hat{U}\right|\right)$$

where $\hat{U}$, M_0 and M are as defined above.

To obtain Zhou's model estimates, it thus suffices to apply the methodology proposed by Beaulieu et al. (2003) in making separate draws based on a U distribution instead of the fully specified W distribution.

When applied to our three families of funds (*Ferique*, *Talvest* and *TD*), this methodology does not allow us to reject the null hypothesis for the existence of the separation theorem at the 99% confidence level. For each of the three tests, we had recourse to 9,999 simulations based on multivariate Student distributions. (See panel Zhou (1993) test in Table 6 for a detailed presentation of the empirical results.)

VI Conclusion

We have provided new elements of response to the asset-allocation puzzle posed by Canner et al. (1997). We first present a careful verification of the reliability of the risk-tolerance measurement used by the authors and obtained a positive result. Our methodology for evaluating the rationality of the portfolios choices made by individual investors is based on Elton and Gruber (2000) study. We therefore tested for the existence of the separation theorem, based on data reflecting all possible portfolio selections for investors with a negative empirical relation between the bonds/stocks ratio and the risk-tolerance index. This test is carried out in two stages: the first consists in checking for the ellipticality of the returns observed on the market, whereas the second involves a test of the Black's CAPM with possibly non-Gaussian error terms and

assuming an environment with no risk free asset and unrestricted short selling. The results obtained favour elliptical returns and confirm the Black-CAPM hypothesis. We then conclude that the asset allocation puzzle does not exist in our data set.

Finally, our results very likely prove that the investors in our data base are not squeezed by constraints on short selling. The fact that very few of them hold negative proportions of assets is more a reflection of personal choice than of a tight constraint.

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Table 1
Descriptive Statistics of Data

	Average	Standard Deviation
Age	54	13
Income ($)	48,034	52,390
Amount in portfolio ($)	91,857	153,494
Income assets objective	42%	31%
Growth assets objective	49%	29%
Speculative assets objective	9%	18%
Weight in stocks	57%	38%
Weight in bonds	38%	40%
Weight in liquidities	5%	7%
Bonds/stocks ratio	0.77	2.52
Excellent knowledge	4%	20%
Good knowledge	34%	47%
Acceptable knowledge	58%	49%
No knowledge	4%	20%
Pension fund account	67%	47%
Margin account	15%	36%
Margin account and Pension fund account	2%	14%

Table 2
Distribution of Total Net Assets of the Investors

Total net assets (in $1,000)	Less than 25	25 to 50	50 to 100	100 to 250	250 to 500	More than 500
Proportion	4%	5%	13%	37%	27%	14%

Table 3
Distribution of Clients according to Financial Advisors

	Proportion	Description
Advisor 1	27%	Young advisor, 3 years of experience
Advisor 2	54%	10 years of experience, with a large clientele
Advisors 3 and 4	19%	15 years of experience, with a wealthy clientele

Table 4
Results from Regressions Using Different Measurements of Risk Tolerance

Independent Variables	Coefficient (*statistic t*)		
	Direct Measurement (R1)	Indirect Measurement (R2)	Indirect Measurement (R3)
Constant	0.9846** (43.4066)	0.7761** (4.9701)	2.1340** (49.0278)
T(Dir)	-1.0587** (-105.3345)		
T(Ind)		-0.8896** (-5.7272)	
T(Obs)			-2.2397** (-67.8738)
Age	0.0003 (0.9202)	0.0045** (2.6350)	0.0006 (1.3839)
Income ($1,000)	0.0001 (1.0587)	-0.0006 (-1.2839)	0.0001 (0.8311)
Asset (0 to 25)	-0.0022 (-0.1227)	0.0868 (0.9273)	-0.0015 (-0.0563)
Asset (25 to 50)	-0.0100 (-0.6572)	0.0450 (0.5663)	-0.0127 (-0.5515)
Asset (50 to 100)	0.0014 (0.1200)	0.0591 (0.9899)	0.0034 (0.1968)
Asset (250 to 500)	0.0011 (0.1226)	0.0124 (0.2692)	0.0010 (0.0747)
Asset (more than 500)	0.0031 (0.2540)	0.0941 (1.4908)	0.0068 (0.3696)
Amount of portfolio ($1,000)	-0.0001** (-2.9735)	-0.0006** (-3.4170)	-0.0002** (-3.2080)
Financial advisor 1	-0.0037 (-0.4145)	-0.1724** (-3.6130)	-0.0130 (-0.9695)
Financial advisors 3&4	-0.0091 (-0.9051)	-0.0188 (-0.3572)	-0.0151 (-0.9905)
Pension fund account	-0.0199* (-2.5246)	0.0154 (0.3736)	-0.0293* (-2.4509)
Margin account	0.0206 (1.9192)	-0.1037 (-1.8626)	0.0273 (1.6763)
Excellent knowledge	0.0033 (0.1818)	-0.2019* (-2.1493)	-0.0002 (-0.0058)
Acceptable knowledge	0.0034 (0.4540)	-0.0605 (-1.5675)	0.0033 (0.2911)
No knowledge	-0.0151 (-0.8407)	-0.0363 (-0.3864)	-0.0230 (-0.8442)
	$R^2 = 97.40\%$	$R^2 = 28.80\%$	$R^2 = 94\%$
Number of observations	405	405	405
Regression F statistic (16, 388)	904.52	9.79	379.74
Slope F statistic (16, 388)	1660.52	12.24	395.43

* significant at 5%; ** significant at 1%

The following dummy variables are part of the constant: Financial advisor 2, Good knowledge, Margin account **and** Pension fund account, and Asset (100 to 250). The slope F statistic allows us to test if $\hat{\beta}_0$ and $\hat{\beta}_2$ are jointly and significantly different from zero. The tabulated Fisher statistic F(16,388) at a confidence level of 95% stands at 1.6696.

Table 5
Statistics on Returns Used in Our Separation Test

	Average	Standard Deviation	Skewness	Kurtosis
Ferique – Equities	0.0134	0.0395	−0.9924	5.9434
Ferique – Bonds	0.0070	0.0133	0.3540	2.7708
Ferique – Money	0.0040	0.0011	0.8953	3.9003
Ferique – Balanced	0.0117	0.0282	−0.8186	5.7848
Talvest – Equities	0.0137	0.0483	−0.6383	7.5227
Talvest – Bonds	0.0065	0.0141	0.3682	3.2926
Talvest – Money	0.0036	0.0011	0.7644	3.3844
Talvest – Balanced	0.0101	0.0302	−0.7545	5.1402
TD – Equities	0.0146	0.0458	−1.0841	7.3410
TD – Bonds	0.0094	0.0158	0.2713	2.7007
TD – Money	0.0036	0.0009	0.6045	2.7264
TD – Balanced	0.0096	0.0234	−1.3929	7.8637

Table 6
Results of the BCAPM Test for the Various Funds Studied

	Beaulieu et al. (2003) test				Zhou (1993) test			
	LR	DF	γ	p-value	LR	DF	γ	p-value
Ferique Funds	0.455	10	0.398	0.267	0.449	10	0.397	0.281
Talvest Funds	1.034	15	0.362	0.249	1.016	15	0.359	0.257
TD Funds	5.586	8	0.350	0.156	5.534	8	0.353	0.153

DF: Number of degrees of freedom of the Student distribution

LR: Likelihood ratio

p-value: When p-value is greater than 0.01, we do not reject the Black-CAPM at 99% level of confidence.

γ : Return of the zero covariance portfolio

Table 7
Correlation between Returns on the Funds Considered

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
Ferique – Equities (1)	100	49.17	12.03	92.99	84.14	35.91	0.77	88.93	87.24	31.94	2.20	87.58
Ferique – Bonds (2)	–	100	22.46	61.34	46.14	87.77	12.72	50.61	40.54	84.43	17.61	59.11
Ferique – Money (3)	–	–	100	10.15	8.49	14.56	85.91	9.09	7.21	17.15	87.81	14.84
Ferique – Balanced (4)	–	–	–	100	82.92	46.93	−3.97	92.62	85.30	43.71	−2.17	88.29
Talvest – Equities (5)	–	–	–	–	100	40.74	−0.54	89.24	90.98	41.51	1.22	80.12
Talvest – Bonds (6)	–	–	–	–	–	100	15.32	46.38	31.65	96.00	18.23	55.83
Talvest – Money (7)	–	–	–	–	–	–	100	−0.30	−4.53	17.27	92.8	4.89
Talvest – Balanced (8)	–	–	–	–	–	–	–	100	90.20	44.57	−0.32	89.27
TD – Equities (9)	–	–	–	–	–	–	–	–	100	32.40	−4.06	82.62
TD – Bonds (10)	–	–	–	–	–	–	–	–	–	100	19.39	53.35
TD – Money (11)	–	–	–	–	–	–	–	–	–	–	100	5.55
TD – Balanced (12)	–	–	–	–	–	–	–	–	–	–	–	100

Figure 1
Short-selling and risk free asset:
Impact on optimal asset allocation

Figure 1.1

Figure 1.2
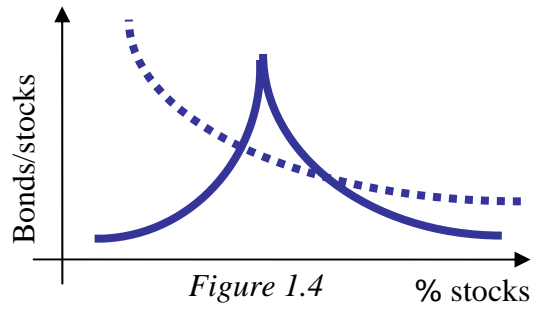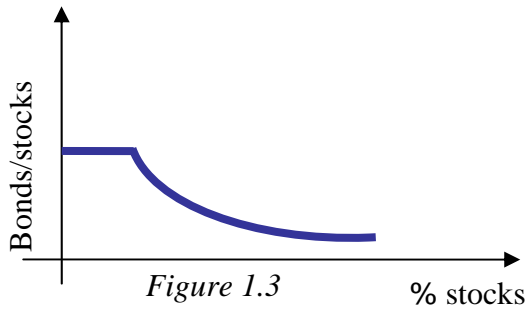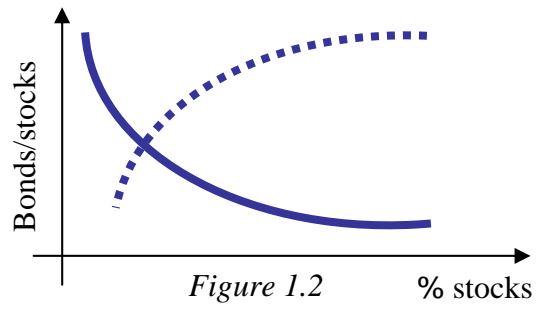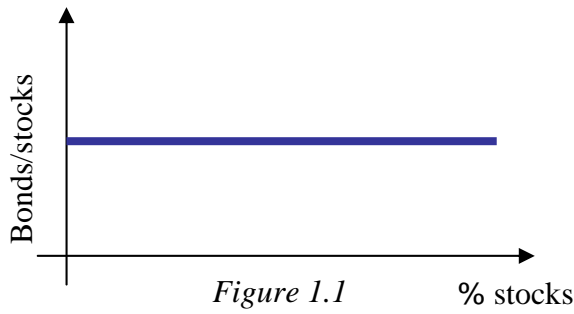
Figure 1.3

Figure 1.4

Figure 1.1   Possibility of short-selling and presence of risk- free asset
Figure 1.2   Possibility of short-selling and absence of risk- free asset
Figure 1.3   No short-selling and presence of risk- free asset
Figure 1.4   No short-selling and absence of risk- free asset

Figure 2
Variation of bonds/stocks ratio observed, as based on
proportion of portfolio invested in stocks

Proportion of portfolio invested in stocks