Comparative Ross Risk Aversion in the Presence of Quadrant Dependent Risks

Georges Dionne

Canada Research Chair in Risk Management, HEC Montréal, CIRRELT, and CIRPÉE,

Canada

Email: georges.dionne@hec.ca

Jingyuan Li

Department of Finance and Insurance

Lingnan University

8 Castle Peak Road, Tuen Mun, Hong Kong

Email: jingyuanli@ln.edu.hk

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Abstract

This paper studies comparative risk aversion between risk averse agents in the presence of a background risk. Although the literature covers this question extensively, our contribution differs from most of the literature in two respects. First, background risk does not need to be additive or multiplicative. Second, the two risks are not necessary mean independent, and may be quadrant dependent. We show that our order of cross Ross risk aversion is equivalent to that of partial risk premium, while our index of decreasing cross Ross risk aversion is equivalent to that of a decreasing partial risk premium. These results generalize the comparative risk aversion model developed by Ross (1981) for mean independent risks. Finally, we show that decreasing cross Ross risk aversion gives rise to the utility function family belonging to the class of n-switch utility functions.

Key words: Comparative cross Ross risk aversion, Quadrant dependent background risk, Partial risk premium, Decreasing cross Ross risk aversion, n-switch utility functions.

JEL classification: D81.

1 Introduction

Arrow (1965) and Pratt (1964) propose an important theorem stating that risk aversion comparisons using risk premia and measures of risk aversion always give the same result. Ross (1981) shows that when an agent faces more than one risk, Arrow-Pratt measures are not strong enough to support the plausible association between absolute risk aversion and the size of the risk premium. He proposes a stronger ordering called Ross risk aversion. Several studies extend Ross' results. Most papers generalize them to higher-orders of risk aversion for univariate utility functions (see Modica and Scarsini, 2005; Jindapon and Neilson, 2007; Li, 2009; Denuit and Eeckhoudt, 2010a). This paper provides another direction to this line of research.

There is growing concern about risk attitudes of bivariate utility function in the literature (see Courbage, 2001; Bleichrodt *et al.*, 2003; Eeckhoudt *et al.*, 2007; Courbage and Rey, 2007; Menegatti, 2009 a,b; Denuit and Eeckhoudt, 2010b; Li, 2011; Denuit *et al.*, 2011a). To our knowledge, these studies do not analyze comparative risk aversion. The first paper that looks at preservation of "more risk averse" with general multivariate preferences and background risk is Nachman (1982). However, in his setting the background risk is independent. Pratt (1988) also considers the comparison of risk aversion both with and without the presence of an independent background risk using a two-argument utility function.

This paper examines comparative Ross risk aversion in the setting of a positive quadrant dependent (PQD, or negative quadrant dependent, NQD) background risk[1]. First, we extend Finkelshtain *et al.*'s (1999) research by analyzing comparative risk aversion in a slightly different context. Then we introduce the notion of cross Ross risk aversion and show that more cross Ross risk aversion is associated with a higher partial risk premium in the presence of a PQD (or NQD) background risk. Hence, we demonstrate that the index of cross Ross risk aversion is equivalent to the order of partial risk premium. We also propose the concept of decreasing cross Ross risk aversion and derive necessary and sufficient conditions for obtaining an equivalence between decreasing cross Ross risk aversion and decreasing partial risk premium for a PQD (or NQD) background risk. We apply this result to examine the effects of changes in wealth and financial background risk on the intensity of risk aversion. Finally, we show that specific

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[1] The concept of quadrant dependence was introduced by Lehmann (1966). Portfolio selection problems with quadrant dependence have been explored by Pellerey and Semeraro (2005) and Dachraoui and Dionne (2007), among others.

assumptions about the behavior of the decreasing cross Ross risk aversion gives rise to the utility function form that belongs to the class of n-switch utility functions (Abbas and Bell, 2011).

Our paper is organized as follows. In Section 2, we review some concepts of dependence. In Section 3, we consider necessary and sufficient conditions for risk aversion to one risk in the presence of a PQD (or NQD) background risk. Section 4 offers the necessary and sufficient conditions for comparing two agents' attitudes towards risk with different utility functions. Section 5 considers the same agent's attitude at different wealth levels under a PQD (or NQD) background risk. Section 6 applies our results to financial background risks. Section 7 relates decreasing cross Ross risk aversion to the n-switch independence property. Section 8 concludes the paper.

2 Review of some concepts of dependence

Let $F(x, y)$ denote the joint distribution and $F_X(x)$ and $F_Y(y)$ the marginal distribution of $\tilde{x}$ and $\tilde{y}$. Ross (1981) consider the following relationship between $\tilde{x}$ and $\tilde{y}$.

Definition 2.1 *(Ross, 1981) $(\tilde{x}, \tilde{y})$ is mean independent if $E[\tilde{y}|\tilde{x} = x] = E(\tilde{y})$ for all x.*

Mean independence is a stronger restriction than uncorrelatedness. However, it is weaker than independence. Lehmann (1966) introduced the following general concept to investigate positive dependence.

Definition 2.2 *(Lehmann, 1966) $(\tilde{x}, \tilde{y})$ is positively quadrant dependent, written $PQD(\tilde{x}, \tilde{y})$, if*

$$F(x, y) \geq F_X(x)F_Y(y) \qquad for \ all \ x, y. \tag{1}$$

(1) can be rewritten as

$$F_X(x|\tilde{y} \leq y) \geq F_X(x). \tag{2}$$

$\tilde{x}$ and $\tilde{y}$ are negative quadrant dependent, written $NQD(\tilde{x}, \tilde{y})$, if the above inequalities hold with the inequality sign reversed. Lehmann interpreted (1) as follows: "knowledge of $\tilde{y}$ being small increases the probability of $\tilde{x}$ being small". In the economic literature (see for example Gollier, 2007), positive (or negative) quadrant dependence is related to first-order stochastic dominance: $F_X(x)$ first-order dominates (or is dominated by) $F_X(x|\tilde{y} \leq y)$ under $PQD(\tilde{x}, \tilde{y})$ $(NQD(\tilde{x}, \tilde{y}))$. Pellerey and Semeraro (2005) assert that a large subset of the multivariate elliptical distribution class is PQD. For more examples, see Joe (1997).

We now propose relationships between the three following definitions: $E[\tilde{y}|\tilde{x} = x] = E(\tilde{y})$, $E[\tilde{y}|\tilde{x} = x]$ is non-decreasing in x (Finkelshtain *et al.*, 1999) and $PQD(\tilde{x}, \tilde{y})$.

Proposition 2.3

$$E[\tilde{y}|\tilde{x} = x] = E(\tilde{y}) \ for \ all \ x \Rightarrow E[\tilde{y}|\tilde{x} = x] \ is \ \ non\text{-}decreasing \ in \ x \Rightarrow PQD(\tilde{x}, \tilde{y}). \quad (3)$$

Proof See the Appendix.

3 Risk aversion with two risks

We consider an economic agent whose preference for wealth, $\tilde{w}$, and a variable, $\tilde{y}$, can be represented by a bivariate model of expected utility. We let $u(w, y)$ denote the utility function, and let $u_1(w, y)$ denote $\frac{\partial u}{\partial w}$ and $u_2(w, y)$ denote $\frac{\partial u}{\partial y}$, and follow the same subscript convention for the functions $u_{11}(w, y)$ and $u_{12}(w, y)$ and so on, and assume that the partial derivatives required for any above definition all exist and are continuous.

Let us consider the following definition of risk aversion proposed by Finkelshtain *et al.* (1999).

Definition 3.1 *(Finkelshtain, Kella and Scarsini, 1999) An agent is risk averse in zero-mean risk $\tilde{x}$ with $(\tilde{x}, \tilde{y})$ if*

$$Eu(w + \tilde{x}, \tilde{y}) \le Eu(w + E\tilde{x}, \tilde{y}) \quad (4)$$

for all initial wealth w.

Finkelshtain *et al.* (1999) provide the following necessary and sufficient condition on u for obtaining risk aversion to one risk in the presence of a background risk.

Proposition 3.2 *(Finkelshtain, Kella and Scarsini, 1999) The following statements are equivalent:*

(i) For $\forall w$ and every zero-mean risk $\tilde{x}$ such that $E[\tilde{y}|\tilde{x} = x]$ is non-decreasing in x, inequality (4) holds;

(ii) u is submodular (i.e., $u(x \lor y) + u(x \land y) \le u(x) + u(y)$ for all $x, y \in R^2$) and concave in its first argument.

We now propose an alternative condition on u to obtain risk aversion in the presence of $PQD(\tilde{x}, \tilde{y})$:

Proposition 3.3 *The following statements are equivalent:*

(i) For $\forall w$ and every $\tilde{x}$ with $PQD(\tilde{x}, \tilde{y})$, inequality (4) holds;

(ii) $u_{11} \leq 0$ and $u_{12} \leq 0$.

Proof See the Appendix.

The interpretation of the sign of the u_{12} goes back to De Finetti (1952) and has been studied and extended by Epstein and Tanny (1980); Richard (1975); Scarsini (1988) and Eeckhoudt *et al.* (2007). For example, Eeckhoudt *et al.* (2007) show that $u_{12} \leq 0$ is necessary and sufficient for an agent to be "correlation averse," meaning that a higher level of the second argument mitigates the detrimental effect of a reduction in the first argument. Agents are correlation averse if they always prefer a 50-50 gamble of a loss in x or a loss in y over another 50-50 gamble offering a loss in both x and y.

Propositions 3.2 and 3.3 each have their comparative advantages. More specifically, Proposition 3.2, contrary to Proposition 3.3, does not require that any of the utility function's partial derivatives be continuous. However, regarding applications, differentiability is often a natural requirement.

Proposition 3.3 shows that an agent with both risk aversion (concavity) in its first argument and correlation aversion dislikes a risk in the presence of a PQD background risk. We want to quantify this effect. This can be done by evaluating the maximum amount of money that this agent is ready to pay to escape one component of the bivariate risk in the presence of the other. Chalfant and Finkelshtain (1993) introduced the following idea into the economics literature.

Definition 3.4 *(Chalfant and Finkelshtain, (1993)) For u and v, the partial risk premia π_u and π_v in $\tilde{x}$ for $(\tilde{x}, \tilde{y})$ is defined as*

$$Eu(w + \tilde{x}, \tilde{y}) = Eu(w - \pi_u + E\tilde{x}, \tilde{y}) \tag{5}$$

and

$$Ev(w + \tilde{x}, \tilde{y}) = Ev(w - \pi_v + E\tilde{x}, \tilde{y}). \tag{6}$$

From Proposition 3.3 we know that $u_{11} \leq 0$ and $u_{12} \leq 0$ ($v_{11} \leq 0$ and $v_{12} \leq 0$) if and only if $\pi_u \geq 0$ ($\pi_v \geq 0$) for any risk $\tilde{x}$ with $PQD(\tilde{x}, \tilde{y})$.

4 Comparative cross risk attitudes

The partial risk premia π_u and π_v are the maximal monetary amounts individuals u and v are willing to pay for removing one risk in the presence of a second risk. We derive necessary and sufficient conditions for comparative partial risk premia in the presence of PQD background risk. Extension of the analysis to NQD background risk is discussed later. Let us introduce two definitions of comparative risk aversion motivated by Ross (1981). The following definition uses $-\frac{u_{12}(w,y)}{u_1(w,y)}$ and $-\frac{v_{12}(w,y)}{v_1(w,y)}$ as local measures of correlation aversion.

Definition u is more cross Ross risk averse than v if and only if there exists $\lambda_1, \lambda_2 > 0$ such that for all w, y and y'

$$\frac{u_{12}(w,y)}{v_{12}(w,y)} \geq \lambda_1 \geq \frac{u_1(w,y')}{v_1(w,y')} \tag{7}$$

and

$$\frac{u_{11}(w,y)}{v_{11}(w,y)} \geq \lambda_2 \geq \frac{u_1(w,y')}{v_1(w,y')}. \tag{8}$$

When $u(w,y) = U(w+y)$ in (7) and (8), we obtain the definition of comparative Ross risk aversion for mean independent risks. However, we are interested in comparisons when the agents face two dependent risks which is more general than mean independence. The following proposition provides an equivalent comparison between risk aversion and partial risk premium in the presence of PQD background risks.

Proposition 4.1 *For u, v with $u_1 > 0$, $v_1 > 0$, $v_{11} < 0$, $u_{11} < 0$, $u_{12} < 0$ and $v_{12} < 0$, the following three conditions are equivalent:*

(i) u is more cross Ross risk averse than v.

(ii) There exists $\phi : R \times R \to R$ with $\phi_1 \leq 0$, $\phi_{12} \leq 0$ and $\phi_{11} \leq 0$, and $\lambda > 0$ such that $u = \lambda v + \phi$.

(iii) $\pi_u \geq \pi_v$ for $\forall$ w and $\tilde{x}$ with $PQD(\tilde{x}, \tilde{y})$.

Proof See the Appendix.

When an agent faces a PQD background risk, the cross Ross risk aversion definition establishes an unambiguous relation between more risk version and a higher willingness to pay for insurance. Hence, the cross Ross measure of absolute risk aversion is in line with our intuition in this partial insurance economic problem. Bacause, as mentioned in the preceding section,

$u_{12} \leq 0$ is necessary and sufficient for correlation aversion, the above proposition introduces $-\frac{u_{12}}{u_1}$ as the local measure of correlation aversion.

Proposition 4.1 introduces two extensions of Ross. First, we generalize Ross by replacing the additive utility function by a general bivariate utility function. Second, we consider dependent risks. Suppose at this stage that we maintain Ross assumption that $E[\tilde{y}|\tilde{x}=x]$ is independent of x. It is easy to demonstrate the following proposition in that context:

Conjecture 4.2 *For u, v with $u_1 > 0$, $v_1 > 0$, $v_{11} < 0$ and $u_{11} < 0$, the following three conditions are equivalent:*

(i) There exists $\lambda > 0$ such that for all (w, y): $\frac{u_{11}(w,y)}{v_{11}(w,y)} \geq \lambda \geq \frac{u_1(w,y')}{v_1(w,y')}$;

(ii) There exists $\lambda > 0$ and $\phi : R \times R \to R$ with $\phi_1 \leq 0$ and $\phi_{11} \leq 0$ such that $u = \lambda v + \phi$;

(iii) $\pi_u \geq \pi_v$ for $\forall$ $(\tilde{x}, \tilde{y})$ such that $E[\tilde{y}|\tilde{x}=x]$ is independent of x.

In other words, Ross' result is easily extended to the bivariate case. Observe that in this conjecture, we do not need to know anything about cross-derivatives. This means that cross-derivatives are useful only to take PQD into account. This could be made clearer with the following polar conjecture:

Conjecture 4.3 *For u, v with $u_1 > 0$, $v_1 > 0$, $v_{12} < 0$ and $u_{12} < 0$, the following three conditions are equivalent:*

(i) There exists $\lambda > 0$ such that for all (w, y): $\frac{u_{12}(w,y)}{v_{12}(w,y)} \geq \lambda \geq \frac{u_1(w,y')}{v_1(w,y')}$;

(ii) There exists $\lambda > 0$ and $\phi : R \times R \to R$ with $\phi_1 \leq 0$ and $\phi_{12} \leq 0$ such that $u = \lambda v + \phi$;

(iii) $\pi_u \geq \pi_v$ for $\forall$ $(\tilde{x}, \tilde{y})$ such that $\tilde{y}|\tilde{x}=x$ is degenerated and non-decreasing in x.

Proposition 4.1 in this paper combines these two conjectures in a single proposition by linking PQD to the sign of the cross-derivative of ϕ.

5 Decreasing cross Ross risk aversion with respect to wealth

In this section, we examine how the partial risk premium for a given risk $\tilde{x}$ is affected by a change in initial wealth w, in the presence of a PQD background risk. Fully differentiating equation (5) with respect to w yields[2]

$$Eu_1(w + \tilde{x}, \tilde{y}) = Eu_1(w + E\tilde{x} - \pi_u, \tilde{y}) - \pi'(w)Eu_1(w + E\tilde{x} - \pi_u, \tilde{y}), \tag{9}$$

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[2] Equation (9) has a univariate counterpart in Eeckhoudt and Kimball (1992).

hence,

$$\pi'(w) = \frac{Eu_1(w + E\tilde{x} - \pi_u, \tilde{y}) - Eu_1(w + \tilde{x}, \tilde{y})}{Eu_1(w + E\tilde{x} - \pi_u, \tilde{y})}. \tag{10}$$

Thus, the partial risk premium is decreasing in wealth if and only if

$$Eh(w + E\tilde{x} - \pi_u, \tilde{y}) \geq Eh(w + \tilde{x}, \tilde{y}), \tag{11}$$

where $h \equiv -u_1$ is defined as minus the partial derivative of function u. Because $h_1 = -u_{11} \geq 0$, condition (11) simply states that the partial risk premium of agent h is larger than the partial risk premium of agent u. From Proposition 4.1, this is true if and only if h is more cross Ross risk averse than u. That is, $\exists \lambda_1, \lambda_2 > 0$, for all w, y and y', such that

$$\frac{h_{12}(w, y)}{u_{12}(w, y)} \geq \lambda_1 \geq \frac{h_1(w, y')}{u_1(w, y')} \tag{12}$$

and

$$\frac{h_{11}(w, y)}{u_{11}(w, y)} \geq \lambda_2 \geq \frac{h_1(w, y')}{u_1(w, y')}, \tag{13}$$

or, equivalently,

$$-\frac{u_{112}(w, y)}{u_{12}(w, y)} \geq \lambda_1 \geq -\frac{u_{11}(w, y')}{u_1(w, y')} \tag{14}$$

and

$$-\frac{u_{111}(w, y)}{u_{11}(w, y)} \geq \lambda_2 \geq -\frac{u_{11}(w, y')}{u_1(w, y')}. \tag{15}$$

We obtain the following proposition:

Proposition 5.1 *For u with $u_1 > 0$, $u_{11} < 0$, $u_{12} < 0$, $u_{111} \geq 0$ and $u_{112} \geq 0$, the following three conditions are equivalent:*

(i) the partial risk premium π_u, associated with any $PQD(\tilde{x}, \tilde{y})$ is decreasing in wealth;

(ii) There exists $\phi : R \times R \to R$ with $\phi_1 \leq 0$, $\phi_{12} \leq 0$ and $\phi_{11} \leq 0$, and $\lambda > 0$ such that $-u_1 = \lambda u + \phi$;

(iii) $\exists \lambda_1, \lambda_2 > 0$, for all w, y and y', such that

$$-\frac{u_{112}(w, y)}{u_{12}(w, y)} \geq \lambda_1 \geq -\frac{u_{11}(w, y')}{u_1(w, y')} \tag{16}$$

and

$$-\frac{u_{111}(w, y)}{u_{11}(w, y)} \geq \lambda_2 \geq -\frac{u_{11}(w, y')}{u_1(w, y')}. \tag{17}$$

The proof of Proposition 5.1 is obtained by using (9) to (15).

Proposition 5.1 introduces $-\frac{u_{112}(w,y)}{u_{11}(w,y)}$ and $-\frac{u_{111}(w,y)}{u_{11}(w,y)}$ as local measurements of cross-prudence and prudence. These local measures of prudence are essentially identical to the measure proposed by Kimball (1990). It is well known that, for the single-risk case, DARA is equivalent to the utility function $-u'(.)$ being more concave than $u(.)$ (see for example, Gollier, 2001). Proposition 5.1 is an extension of this result to bivariate risks under a PQD background risk. An interpretation of the sign of u_{112} is provided by Eeckhoudt *et al.* (2007), who showed that $u_{112} > 0$ is a necessary and sufficient condition for "cross-prudence in its second argument", meaning that a higher level of second argument mitigates the detrimental effect of the monetary risk.

There are economic situations where negative dependence is more pertinent. If $\tilde{x}$ and $\tilde{y}$ are NQD, then $\tilde{x}$ and $-\tilde{y}$ are PQD. We can define $m(x,y) = u(x,-y)$, and Propositions 3.3, 4.1 and 5.1 can be applied to $m(x,y)$ directly.

6 Comparative risk aversion in the presence of a financial background risk

Financial background risk has received much attention in the economics literature. For additive financial background risk, we refer to Doherty and Schlesinger (1983a,b, 1986), Kischka (1988), Eeckhoudt and Kimball (1992), Eeckhoudt and Gollier, (2000), Schlesinger (2000), Gollier (2001), Eeckhoudt *et al.* (2007) and Franke *et al.* (2011). For multiplicative financial background risk, see Franke *et al.* (2006, 2011). In this section, we consider some examples to illustrate the use of Propositions 4.1 and 5.1 in the framework of additive or multiplicative background risks.

6.1 Additive background risk

First, we show that Proposition 4.1 allows us to extend the results of Ross (1981) for an additive background risk. Note that, for an additive background risk $\tilde{y}$, we have

$$u(w,y) = U(w+y) \tag{18}$$

and

$$v(w,y) = V(w+y). \tag{19}$$

Here w can be interpreted as the random wealth of an agent and y as a random increment to wealth, i.e., random income or financial portfolio.

Given that

$$u_1 = U' \quad , \quad u_{11} = u_{12} = U'' \quad and \quad u_{111} = u_{112} = U''' \tag{20}$$

and

$$v_1 = V' \quad , \quad v_{11} = v_{12} = V'' \quad and \quad v_{111} = v_{112} = V''', \tag{21}$$

Ross (1981) proposed the following results.

Proposition 6.1 *(Ross (1981, Theorem 3)) For $u(w,y) = U(w+y)$, $v(w,y) = V(w+y)$ with $U' > 0$, $V' > 0$, $U'' < 0$ and $V'' < 0$, the following two conditions are equivalent:*

(i) $\exists \lambda > 0$

$$\frac{U''(w+y)}{V''(w+y)} \geq \lambda \geq \frac{U'(w+y')}{V'(w+y')} \quad for \ \ all \ w \ , y \ and \ y'. \tag{22}$$

(ii) $\pi_u \geq \pi_v$ for $\forall \ w$, any zero-mean risk $\tilde{x}$ and $\tilde{y}$ with $E[\tilde{x}|\tilde{y} = y] = E\tilde{x} = 0$.

Proposition 6.2 *(Ross (1981, Theorem 4)) For $u(w,y) = U(w+y)$, with $U' > 0$, $U'' < 0$ and $U''' > 0$, the partial risk premium associated with any zero-mean risk $\tilde{x}$ with $E[\tilde{x}|\tilde{y} = y] = 0$ is decreasing in wealth if and only if, $\exists \lambda > 0$, for all w, y and y',*

$$-\frac{U'''(w+y)}{U''(w+y)} \geq \lambda \geq -\frac{U''(w+y')}{U'(w+y')} \tag{23}$$

We now show that Propositions 4.1 and 5.1 generalize Ross' conditions.

Conditions (7) and (8) imply

$$\frac{U''(w+y)}{V''(w+y)} \geq \lambda \geq \frac{U'(w+y')}{V'(w+y')} \quad for \ \ all \ w \ , y \ and \ y'. \tag{24}$$

Proposition 4.1, (20), (21) and (24) immediately entail the following result.

Corollary 6.3 *For $u(w,y) = U(w+y)$, $v(w,y) = V(w+y)$ with $U' > 0$, $V' > 0$, $U'' < 0$ and $V'' < 0$, the following two conditions are equivalent:*

(i) $\exists \lambda > 0$

$$\frac{U''(w+y)}{V''(w+y)} \geq \lambda \geq \frac{U'(w+y')}{V'(w+y')} \quad for \ \ all \ w \ , y \ and \ y'. \tag{25}$$

(ii) $\pi_u \geq \pi_v$ for $\forall \ w$ and $PQD(\tilde{x}, \tilde{y})$.

Conditions (16) and (17) imply, for all w, y and y',

$$-\frac{U'''(w+y)}{U''(w+y)} \geq \lambda \geq -\frac{U''(w+y')}{U'(w+y')} \tag{26}$$

From Proposition 5.1, (20), (21) and (26), we obtain the following corollary:

Corollary 6.4 *For $u(w,y) = U(w+y)$, with $U' > 0$, $U'' < 0$ and $U''' > 0$, the following two conditions are equivalent:*

(i) the partial risk premium associated with any $PQD(\tilde{x}, \tilde{y})$ is decreasing in wealth.

(ii) $\exists \lambda > 0$, for all w, y and y',

$$-\frac{U'''(w+y)}{U''(w+y)} \geq \lambda \geq -\frac{U''(w+y')}{U'(w+y')} \tag{27}$$

In Corollary 6.4, the condition for decreasing risk premia under PQD risks is equivalent to that for a first-order stochastic dominance (FSD) improvement in an independent background risk to decrease the risk premium, as shown by Eeckhoudt *et al.* (1996).

6.2 Multiplicative background risk

For a multiplicative background risk $\tilde{y}$, we have

$$u(w, y) = U(wy) \tag{28}$$

and

$$v(w, y) = V(wy). \tag{29}$$

Here w may represent the random wealth invested in a risky asset and y may represent a multiplicative random shock on random wealth, like a variation of random interest rate.

Because

$$u_1 = yU', \quad u_{11} = y^2 U'', \quad u_{12} = U' + wyU'', \quad u_{111} = y^3 U''' \quad and \quad u_{112} = 2yU'' + wy^2 U''' \tag{30}$$

and

$$v_1 = yV', \quad v_{11} = y^2 V'', \quad v_{12} = V' + wyV'', \quad v_{111} = y^3 V''' \quad and \quad v_{112} = 2yV'' + wy^2 V'''. \tag{31}$$

Conditions (7) and (8) imply, $\exists \lambda_1, \lambda_2 > 0$, for all w, y and y',

$$\frac{U'(wy) + wyU''(wy)}{V'(wy) + wyV''(wy)} \geq \lambda_1 \geq \frac{U'(wy')}{V'(wy')} \tag{32}$$

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and

$$\frac{U''(wy)}{V''(wy)} \geq \lambda_2 \geq \frac{U'(wy')}{V'(wy')}. \tag{33}$$

Then, from Proposition 4.1, (53), (54), (57) and (33), we obtain

Corollary 6.5 *For* $u(w,y) = U(wy)$, $v(w,y) = V(wy)$ *with* $U' > 0$, $V' > 0$, $U'' < 0$ *and* $V'' < 0$, *the following two conditions are equivalent:*

(i) $\exists \lambda_1, \lambda_2 > 0$, *for all* w, y *and* y',

$$\frac{U'(wy) + wyU''(wy)}{V'(wy) + wyV''(wy)} \geq \lambda_1 \geq \frac{U'(wy')}{V'(wy')} \tag{34}$$

and

$$\frac{U''(wy)}{V''(wy)} \geq \lambda_2 \geq \frac{U'(wy')}{V'(wy')}. \tag{35}$$

(ii) $\pi_u \geq \pi_v$ *for* $\forall$ w *and* $PQD(\tilde{x}, \tilde{y})$.

Because

$$\frac{U'(wy) + wyU''(wy)}{V'(wy) + wyV''(wy)} \tag{36}$$

$$= \frac{U''(wy)(\frac{U'(wy)}{U''(wy)} + wy)}{V''(wy)(\frac{V'(wy)}{V''(wy)} + wy)}$$

$$= \frac{U''(wy)(wy - \frac{1}{RA_U(wy)})}{V''(wy)(wy - \frac{1}{RA_V(wy)})},$$

where $RA_U(wy) = -\frac{U''(wy)}{U'(wy)}$ and $RA_V(wy) = -\frac{V''(wy)}{V'(wy)}$ are indices of absolute risk aversion. We can obtain a more concise sufficient condition from Corollary 6.5.

Corollary 6.6 *For* $u(w,y) = U(wy)$, $v(w,y) = V(wy)$ *with* $w > 0$, $\tilde{y} > 0$ *almost surely,* $U' > 0$, $V' > 0$, $U'' < 0$ *and* $V'' < 0$, *If* $\exists \lambda > 0$, *for all* w,y *and* y',

$$\frac{U''(wy)}{V''(wy)} \geq \lambda \geq \frac{U'(wy')}{V'(wy')}, \tag{37}$$

then $\pi_u \geq \pi_v$ *for* $\forall$ w *and* $PQD(\tilde{x}, \tilde{y})$.

Proof From Corollary 6.5 and (36), we know that for all w,y and y',

$$\frac{U''(wy)}{V''(wy)} \geq \lambda \geq \frac{U'(wy')}{V'(wy')} \tag{38}$$

and $RA_U(wy) \geq RA_V(wy)$ imply that $\pi_u \geq \pi_v$ for $\forall$ w and $PQD(\tilde{x}, \tilde{y})$. Using the fact that "$U$ is more Ross risk averse than $V \Rightarrow RA_U(wy) \geq RA_V(wy)$", we obtain the result. Q.E.D.

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Corollary 6.6 states that "more Ross risk aversion" is a sufficient condition to order the partial risk premium in the presence of PQD multiplicative background risk.

From Proposition 5.1, we obtain

Corollary 6.7 *For $u(w,y) = U(wy)$, with $U' > 0$, $U'' < 0$ and $U''' > 0$, the partial risk premium associated with any $PQD(\tilde{x}, \tilde{y})$ is decreasing in wealth if and only if, $\exists \lambda_1, \lambda_2 > 0$, for all w,y and y',*

$$-\frac{2yU''(wy) + wy^2U'''(wy)}{U'(wy) + wyU''(wy)} \geq \lambda_1 \geq -\frac{y'U''(wy')}{U'(wy')} \tag{39}$$

and

$$-\frac{yU'''(wy)}{U''(wy)} \geq \lambda_2 \geq -\frac{y'U''(wy')}{U'(wy')}. \tag{40}$$

Because

$$-\frac{2yU''(wy) + wy^2U'''(wy)}{U'(wy) + wyU''(wy)} \tag{41}$$

$$= -\frac{yU'''(wy)(2\frac{U''(wy)}{U'''(wy)} + wy)}{U''(wy)(\frac{U'(wy)}{U''(wy)} + wy)}$$

$$= -\frac{yU'''(wy)(wy - 2\frac{1}{P_U(wy)})}{U''(wy)(wy - \frac{1}{RA_U(wy)})},$$

where $P_U(wy) = -\frac{U'''(wy)}{U''(wy)}$ is the index of absolute prudence. We can obtain a shorter sufficient condition from Corollary 6.7 and (41).

Corollary 6.8 *For $u(w,y) = U(wy)$, with $w > 0$, $\tilde{y} > 0$ almost surely, $U' > 0$, $U'' < 0$ and $U''' > 0$, The partial risk premium associated with $PQD(\tilde{x}, \tilde{y})$ is decreasing in wealth if , $\exists \lambda > 0$, for all w, y and y',*

$$-\frac{yU'''(wy)}{U''(wy)} \geq \lambda \geq -\frac{y'U''(wy')}{U'(wy')} \tag{42}$$

and $P_U(wy) \geq 2RA_U(wy)$.

Moreover, (42) can be multiplied by w on both sides to obtain the results in terms of measures of relative risk aversion and relative prudence:

$$-\frac{wyU'''(wy)}{U''(wy)} \geq \lambda \geq -\frac{wy'U''(wy')}{U'(wy')}, \tag{43}$$

which implies "min relative prudence $\geq$ max relative risk aversion". Whereas in the literature, $P_U \geq 2RA_U$ is an important condition for risk vulnerability (see Gollier 2001, p129), Corollary 6.8 shows that $P_U \geq 2RA_U$ is also an important condition for comparative risk aversion in the presence of a PQD multiplicative background risk.

7 Decreasing cross Ross risk aversion and n-switch independence property

Because the conditions derived in Ross (1981) are fairly restrictive upon preference, some readers may regard Ross' results as negative, because no standard utility functions (logarithmic, power, mixture of exponentials) satisfy these conditions. Pratt (1990) suggests that probability distribution restrictions stronger than mean independence may provide more satisfactory comparative statics. In a very different domain, Bell (1988) proposes that agents are likely to be characterized by a utility function satisfying the one-switch rule: there exists at most one critical wealth level at which the decision-maker switches from preferring one alternative to the other. He shows that the linex function (linear plus exponential) is the only relevant utility function family if one adds to the one-switch rule some very reasonable requirements. This utility function has been studied by Bell and Fishburn (2001), Sandvik and Thorlund-Petersen (2010), Abbas and Bell (2011) and Tsetlin and Winkler (2009, 2012). In a recent paper, Denuit *et al.* (2011b) show that Ross' stronger measure of risk aversion gives rise to the linex utility function and therefore they provide not only a utility function family but also some intuitive and convenient properties for Ross' measure.

Abbas and Bell (2011) extend the one-switch independence property to two-attribute utility functions, and propose a new independence assumption based on the one-switch property: n-switch independence (see Tsetlin and Winkler, 2012, for a similar extension).

Definition (Abbas and Bell 2011) For utility function $u(x, y)$, X is n-switch independent of Y if two gambles $\tilde{x}_1$ and $\tilde{x}_2$ can switch in preference at most n times as Y progresses from its lowest to its highest value.

They provide the following propositions:

Proposition 7.1 *(Abbas and Bell 2011) X is one-switch independent of Y if and only if*

$$u(x, y) = g_0(y) + f_1(x)g_1(y) + f_2(x)g_2(y),\tag{44}$$

where $g_1(y)$ has a constant sign, and $g_2(y) = g_1(y)\phi(y)$ for some monotonic function ϕ.

Proposition 7.2 *(Abbas and Bell 2011) If X is n-switch independent of Y, then there exist some functions f_i, g_i such that*

$$u(x, y) = g_0(y) + \sum_{i=1}^{n+1} f_i(x)g_i(y).\tag{45}$$

We now show that the one-switch property of Proposition 7.1 is a consequence of Proposition 5.1. We also argue that (45) is a utility function that satisfies the decreasing cross Ross risk aversion condition proposed in Section 3.

From Proposition 5.1 we know that the partial risk premium π_u, associated with any $PQD(\tilde{x}, \tilde{y})$ is decreasing in wealth, if and only if there exists $\phi : R \times R \to R$ with $\phi_1 \leq 0$, $\phi_{12} \leq 0$ and $\phi_{11} \leq 0$, and $\lambda > 0$ such that

$$-u_1(x, y) = \lambda u(x, y) + \phi(x, y). \tag{46}$$

Solving the above differential equation implies that u is of the form

$$u(x, y) = -\int_{-\infty}^{x} e^{\lambda t}\phi(t, y)dt e^{-\lambda x}. \tag{47}$$

If we take $\phi(x, y) = -H(x)J(y)$ such that $J(y)$ has a constant sign, then we get

$$
\begin{aligned}
u(x, y) &= \int_{-\infty}^{x} e^{\lambda t} H(t)dt e^{-\lambda x} J(y) \tag{48}\\
&= [\frac{1}{\lambda}e^{\lambda x}H(x) - \frac{1}{\lambda}\int_{-\infty}^{x} e^{\lambda t}H'(t)dt]e^{-\lambda x}J(y)\\
&= \frac{1}{\lambda}H(x)J(y) - \frac{1}{\lambda}\int_{-\infty}^{x} e^{\lambda t}H'(t)dt e^{-\lambda x}J(y).
\end{aligned}
$$

Defining $g_1(y) = g_2(y) = \frac{1}{\lambda}J(y)$, $f_1(x) = H(x)$ and $f_2(x) = -\int_{-\infty}^{x} e^{\lambda t}H'(t)dt e^{-\lambda x}$, we recognize the functional form in Proposition 7.1.

Integrating the integral term of (48) by parts again and again, we obtain

$$
\begin{aligned}
u(x, y) &= \sum_{i=1}^{n} e^{\lambda x}\frac{(-1)^{i-1}H^{(i-1)}(x)}{\lambda^i} + \frac{1}{\lambda^n}\int_{-\infty}^{x} e^{\lambda t}(-1)^n H^{(n)}(t)dt]e^{-\lambda x}J(y) \tag{49}\\
&= \sum_{i=1}^{n} J(y)\frac{(-1)^{i-1}H^{(i-1)}(x)}{\lambda^i} + \frac{1}{\lambda^n}\int_{-\infty}^{x} e^{\lambda t}(-1)^n H^{(n)}(t)dt e^{-\lambda x}J(y)\\
&= \sum_{i=1}^{n+1} f_i(x)g_i(y),
\end{aligned}
$$

where $f_i(x) = (-1)^{(i-1)}H^{(i-1)}(x)$ for $i = 1, ..., n$, $f_{n+1}(x) = \int_{-\infty}^{x} e^{\lambda t}(-1)^n H^{(n)}(t)dt e^{-\lambda x}$, $g_i(y) = \frac{1}{\lambda^i}J(y)$ for $i = 1, .., n$ and $g_{n+1}(y) = \frac{1}{\lambda^n}J(y)$. Therefore we obtain the functional form in Proposition 7.2 from decreasing cross Ross risk aversion. Although coming from very different approaches, decreasing cross Ross risk aversion and n-switch independence reach the same functional form. Our result thus provides a connection between decreasing cross Ross risk aversion and n-switch independence.

8 Conclusion

In this paper we consider expected-utility preferences in a bivariate setting. The analysis focuses on PQD random variables. The main contribution is to propose a risk premium for removing one of the risks in the presence of a second dependent risk. To this end, we extend Ross' (1981) contribution by defining the concept of "cross Ross risk aversion." We derive several equivalence theorems relating measures of risk premia with measures of risk aversion. We then consider additive risks and multiplicative risks as two special cases. We also show that the decreasing cross Ross risk aversion assumption about behavior gives rise to the utility function family that belongs to the class of n-switch utility functions. The analysis and the index of risk aversion in this paper may be instrumental in obtaining comparative static predictions in various applications.

9 Appendix

9.1 Proof of Proposition 2.3

It is obvious that $E[\tilde{y}|\tilde{x} = x] = E(\tilde{y})$ implies $E[\tilde{y}|x]$ is non-decreasing in x. We now consider PQD$(\tilde{x}, \tilde{y})$. Cohen *et al.* (1994) introduce the concept of conditionally increasing in sequence:

Definition 9.1 *(Cohen et al. 1994) The random variables $(\tilde{y}, \tilde{x})$ are said to be conditionally increasing in sequence (CIS) if*

$$E[\tilde{y}|\tilde{x} = x] \leq E[\tilde{y}|\tilde{x} = x^*], \tag{50}$$

for $x \leq x^*$.

We know that $E[\tilde{y}|\tilde{x} = x]$ non-decreasing in x implies that $(\tilde{y}, \tilde{x})$ are CIS. From the theorems in Cohen *et al.* (1994, Theorem 2.5) and Joe (1997, Theorem 2.3 (b)), we obtain

$$E[\tilde{y}|\tilde{x} = x] \ is \ non - decreasing \ in \ x \Rightarrow PQD(\tilde{x}, \tilde{y}). \tag{51}$$

Q.E.D.

9.2 Proof of Proposition 3.3

We will use following notations: $d_x F(x, y) = \frac{\partial F(x,y)}{\partial x} dx$, $d_y F(x, y) = \frac{\partial F(x,y)}{\partial y} dy$ and $d_x d_y F(x, y) = \frac{\partial^2 F(x,y)}{\partial x \partial y} dx dy$.

(ii) implies (i): First, we have

$$Eu(w + \tilde{x}, \tilde{y}) - Eu(w + E\tilde{x}, \tilde{y}) \tag{52}$$

$$= \int \int u(w + x, y) d_x d_y F(x, y) - \int u(w + E\tilde{x}, y) d_y F_Y(y)$$

$$\leq \int \int u(w + x, y) d_x d_y F(x, y) - \int \int u(w + x, y) d_x F_X(x) d_y F_Y(y) \quad (because \ u_{11} \leq 0)$$

$$= \int \int u(w + x, y) d_x d_y F(x, y) - \int \int u(w + x, y) d_x d_y H(x, y),$$

where $H(x, y) = F_X(x) F_Y(y)$. From Levy (1974, corollary 4), we know that

$$\int \int u(w + x, y) d_x d_y F(x, y) - \int \int u(w + x, y) d_x d_y H(x, y) \tag{53}$$

$$= \int \int u_{12}(w + x, y)[F(x, y) - H(x, y)] dx dy$$

$$+ \lim_{y \to \infty} \int (H_X(x) - F_X(x)) u_1(w + x, y) dx + \lim_{x \to \infty} \int (H_Y(y) - F_Y(y)) u_2(w + x, y) dy$$

$$= \int \int u_{12}(w + x, y)[F(x, y) - H(x, y)] dx dy \quad (because \ F_X(x) = H_X(x) \ and \ F_Y(y) = H_Y(y))$$

$$= \int \int u_{12}(w + x, y)[F(x, y) - F_X(x) F_Y(y)] dx dy \leq 0 \quad (because \ u_{12} \leq 0 \ and \ PQD(\tilde{x}, \tilde{y})).$$

From the above manipulations, we obtain that $Eu(w + \tilde{x}, \tilde{y}) \leq Eu(w + E\tilde{x}, \tilde{y})$.

(i) implies (ii): We prove this claim by contradictions. Suppose $u_{12}(w, y) > 0$ for some w and y. Because u_{12} is continuous, we have

$$u_{12}(w, y) > 0 \quad for \quad (w, y) \in [m_1, m_2] \times [n_1, n_2]. \tag{54}$$

Let us consider $w_0 \in [m_1, m_2]$ and $\tilde{x} = k\tilde{z}$ with $k > 0$, where $\tilde{z}$ is a zero-mean risk and $(\tilde{z}, \tilde{y})$ is PQD with the joint distribution function $G(z, y)$[3]. Using Taylor expansion of $Eu(w_0 + k\tilde{z}, \tilde{y})$ around w_0, this yields, for any k:

$$Eu(w_0 + k\tilde{z}, \tilde{y}) = E[u(w_0, \tilde{y})] + E[\tilde{z} u_1(w_0, \tilde{y})] k + o(k). \tag{55}$$

Because

$$E\tilde{z} u_1(w_0, \tilde{y}) \tag{56}$$

$$= E\tilde{z} E u_1(w_0, \tilde{y}) + Cov(\tilde{z}, u_1(w_0, \tilde{y}))$$

$$= Cov(\tilde{z}, u_1(w_0, \tilde{y}))$$

$$= \int \int [G(z, y) - G_Z(z) G_Y(y)] dz d_y u_1(w_0, y) \quad (by \ Cuadras \ 2002, \ Theorem \ 1)$$

$$= \int \int [G(z, y) - G_Z(z) G_Y(y)] u_{12}(w_0, y) dz dy.$$

---

[3]Lehmann (1966, Lemma 1) showed that $(\tilde{x}, \tilde{y})$ is PQD $\Rightarrow (r(\tilde{x}), s(\tilde{y}))$ is PQD, for all non-decreasing functions r and s

Then, from (54) we know that, when $k \to 0$, we get $Eu(w_0 + \tilde{x}, \tilde{y}) > Eu(w_0, \tilde{y})$ for $G(z, y)$ such that $G(z, y) - G_Z(z)G_Y(y)$ is positive in domain $[m_1, m_2] \times [n_1, n_2]$ and zero elsewhere. This is a contradiction.

Suppose $u_{11}(w, y) > 0$ for some w and y. Because u_{11} is continuous, we have

$$u_{11}(w, y) > 0 \quad for \quad (w, y) \in [m_1', m_2'] \times [n_1', n_2']. \tag{57}$$

Let us consider $w_0 \in [m_1', m_2']$ and $\tilde{x} = k\tilde{z}$, where $\tilde{z}$ is a zero-mean risk and $(\tilde{z}, \tilde{y})$ are independent. Using Taylor expansion of $Eu(w_0 + k\tilde{z}, \tilde{y})$ around w_0. For any k, this yields

$$Eu(w_0 + k\tilde{z}, \tilde{y}) = E[u(w_0, \tilde{y})] + \frac{1}{2}E[u_{11}(w_0, \tilde{y})]E\tilde{z}^2 k^2 + o(k^2). \tag{58}$$

Then, from (57) we know that, when $k \to 0$, we get $Eu(w_0 + \tilde{x}, \tilde{y}) > Eu(w_0, \tilde{y})$ for $F(x, y)$ such that $F_Y(y)$ has positive support on interval $[n_1', n_2']$. This is a contradiction. Q.E.D.

9.3 Proof of Proposition 4.1

(i) implies (ii): We note that

$$\frac{u_{12}(w, y)}{v_{12}(w, y)} \geq \lambda_1 \geq \frac{u_1(w, y')}{v_1(w, y')} \Leftrightarrow \frac{-u_{12}(w, y)}{-v_{12}(w, y)} \geq \lambda_1 \geq \frac{u_1(w, y')}{v_1(w, y')}. \tag{59}$$

$$\frac{u_{11}(w, y)}{v_{11}(w, y)} \geq \lambda_2 \geq \frac{u_1(w, y')}{v_1(w, y')} \Leftrightarrow \frac{-u_{11}(w, y)}{-v_{11}(w, y)} \geq \lambda_2 \geq \frac{u_1(w, y')}{v_1(w, y')}. \tag{60}$$

Defining $\phi = u - \lambda v$, ,where $\lambda = \min\{\lambda_1, \lambda_2\}$, and differentiating one obtains $\phi_1 = u_1 - \lambda v_1$, $\phi_{12} = u_{12} - \lambda v_{12}$ and $\phi_{11} = u_{11} - \lambda v_{11}$, then (59) and (60) imply that $\phi_1 \leq 0$, $\phi_{12} \leq 0$ and $\phi_{11} \leq 0$.

(ii) implies (iii): From Proposition 3.3, we know that $\phi_{11} \leq 0$, $\phi_{12} \leq 0$ and $(\tilde{x}, \tilde{y})$ is PQD$(\tilde{x}, \tilde{y})$ $\Leftrightarrow E\phi(w + \tilde{x}, \tilde{y}) \leq E\phi(w, \tilde{y})$. We also know that $\phi_1 \leq 0 \Rightarrow \phi(w, y) \leq \phi(w - \pi_v, y)$. The following proof is as in Ross:

$$
\begin{aligned}
Eu(w - \pi_u + E\tilde{x}, \tilde{y}) &= Eu(w + \tilde{x}, \tilde{y}) \\
&= E[\lambda v(w + \tilde{x}, \tilde{y}) + \phi(w + \tilde{x}, \tilde{y})] \\
&= \lambda Ev(w - \pi_v, \tilde{y}) + E\phi(w + \tilde{x}, \tilde{y}) \\
&\leq \lambda Ev(w - \pi_v, \tilde{y}) + E\phi(w, \tilde{y}) \\
&\leq \lambda Ev(w - \pi_v, \tilde{y}) + E\phi(w - \pi_v, \tilde{y}) \\
&= Eu(w - \pi_v + E\tilde{x}, \tilde{y}).
\end{aligned}
\tag{61}
$$

Because $u_1 > 0$, $\pi_u \geq \pi_v$.

(iii) implies (i): We prove this claim by contradictions. Suppose that there exists some w, y and y' such that $\frac{u_{12}(w,y)}{v_{12}(w,y)} < \frac{u_1(w,y')}{v_1(w,y')}$. Because u_1, v_1, u_{12} and v_{12} are continuous, we have

$$\frac{u_{12}(w,y)}{v_{12}(w,y)} < \frac{u_1(w,y')}{v_1(w,y')} \quad for \quad (w,y),(w,y') \in [m_1,m_2] \times [n_1,n_2], \tag{62}$$

which implies

$$\frac{-u_{12}(w,y)}{-v_{12}(w,y)} < \frac{u_1(w,y')}{v_1(w,y')} \quad for \quad (w,y),(w,y') \in [m_1,m_2] \times [n_1,n_2], \tag{63}$$

this implies

$$\frac{v_1(w,y')}{-v_{12}(w,y)} < \frac{u_1(w,y')}{-u_{12}(w,y)} \quad for \quad (w,y),(w,y') \in [m_1,m_2] \times [n_1,n_2]. \tag{64}$$

If $F(x,y)$ is a distribution function such that $F_Y(y)$ has positive support on interval $[n_1,n_2]$, then we have

$$\frac{Ev_1(w,\tilde{y})}{-v_{12}(w,y)} < \frac{Eu_1(w,\tilde{y})}{-u_{12}(w,y)} \quad for \quad (w,y) \in [m_1,m_2] \times [n_1,n_2], \tag{65}$$

which can be written as

$$\frac{u_{12}(w,y)}{Eu_1(w,\tilde{y})} > \frac{v_{12}(w,y)}{Ev_1(w,\tilde{y})} \quad for \quad (w,y) \in [m_1,m_2] \times [n_1,n_2]. \tag{66}$$

Let us consider $w_0 \in [m_1,m_2]$ and $\tilde{x} = k\tilde{z}$ with $k > 0$, where $\tilde{z}$ is a zero-mean risk and $(\tilde{z},\tilde{y})$ is PQD with a distribution function $G(z,y)$. Let $\pi_u(k)$ denote its associated partial risk premium, which is

$$Eu(w_0 + k\tilde{z},\tilde{y}) = Eu(w_0 - \pi_u(k),\tilde{y}). \tag{67}$$

Differentiating the equality above with respect to k yields

$$E\tilde{z}u_1(w_0 + k\tilde{z},\tilde{y}) = -\pi_u'(k)Eu_1(w_0 - \pi_u(k),\tilde{y}). \tag{68}$$

Observing that $\pi_u(0) = 0$, we get

$$
\begin{aligned}
\pi_u'(0) &= -\frac{E\tilde{z}u_1(w_0,\tilde{y})}{Eu_1(w_0,\tilde{y})} \\
&= -\frac{E\tilde{z}Eu_1(w_0,\tilde{y}) + Cov(\tilde{z},u_1(w_0,\tilde{y}))}{Eu_1(w_0,\tilde{y})} \\
&= -\frac{Cov(\tilde{z},u_1(w_0,\tilde{y}))}{Eu_1(w_0,\tilde{y})} \\
&= -\frac{\int\int[G(z,y) - G_Z(z)G_Y(y)]dzd_yu_1(w_0,y)}{Eu_1(w_0,\tilde{y})} \quad (by \ Cuadras \ 2002, \ Theorem \ 1) \\
&= -\int\int[G(z,y) - G_Z(z)G_Y(y)]\frac{u_{12}(w_0,y)}{Eu_1(w_0,\tilde{y})}dzdy
\end{aligned}
\tag{69}
$$

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Similarly, for v we have

$$\pi_v'(0) = -\int\int[G(z,y) - G_Z(z)G_Y(y)]\frac{v_{12}(w_0,y)}{Ev_1(w_0,\tilde{y})}dzdy. \tag{70}$$

Now π_u and π_v can be written in the form of a Taylor expansion around $k=0$:

$$\pi_u(k) = -k\int\int[G(z,y) - G_Z(z)G_Y(y)]\frac{u_{12}(w_0,y)}{Eu_1(w_0,\tilde{y})}dzdy + o(k) \tag{71}$$

and

$$\pi_v(k) = -k\int\int[G(z,y) - G_Z(z)G_Y(y)]\frac{v_{12}(w_0,y)}{Ev_1(w_0,\tilde{y})}dzdy + o(k). \tag{72}$$

Then, from (66) we know that, when $k \to 0$, we get $\pi_u < \pi_v$ for $F(x,y)$ and $G(z,y)$ such that $F_Y(y)$ and $G(z,y) - G_Z(z)G_Y(y)$ have positive supports on domain $[m_1,m_2] \times [n_1,n_2]$. This is a contradiction.

Now let us turn to the other condition. Suppose that there exists some w, y and y' such that $\frac{u_{11}(w,y)}{v_{11}(w,y)} < \frac{u_1(w,y')}{v_1(w,y')}$. Because u_1, v_1, u_{11} and v_{11} are continuous, we have

$$\frac{u_{11}(w,y)}{v_{11}(w,y)} < \frac{u_1(w,y')}{v_1(w,y')} \quad for \quad (w,y),(w,y') \in [m_1',m_2'] \times [n_1',n_2'], \tag{73}$$

which implies

$$\frac{-u_{11}(w,y)}{-v_{11}(w,y)} < \frac{u_1(w,y')}{v_1(w,y')} \quad for \quad (w,y),(w,y') \in [m_1',m_2'] \times [n_1',n_2']. \tag{74}$$

This implies

$$\frac{-u_{11}(w,y)}{u_1(w,y')} < \frac{-v_{11}(w,y)}{v_1(w,y')} \quad for \quad (w,y),(w,y') \in [m_1',m_2'] \times [n_1',n_2']. \tag{75}$$

If $F(x,y)$ is a distribution function such that $F_Y(y)$ has positive support on interval $[n_1',n_2']$, then we have

$$\frac{-Eu_{11}(w,\tilde{y})}{u_1(w,y')} < \frac{-Ev_{11}(w,\tilde{y})}{v_1(w,y')} \quad for \quad (w,y') \in [m_1',m_2'] \times [n_1',n_2'] \tag{76}$$

and

$$\frac{-Eu_{11}(w,\tilde{y})}{Eu_1(w,\tilde{y})} < \frac{-Ev_{11}(w,\tilde{y})}{Ev_1(w,\tilde{y})}. \tag{77}$$

Let us consider $w_0 \in [m_1',m_2']$ and $\tilde{x} = k\tilde{z}$, where $\tilde{z}$ is a zero-mean risk and $\tilde{z}$ and $\tilde{y}$ are independent. Let $\pi_u(k)$ denote its associated partial risk premium, which is defined by

$$Eu(w_0 + k\tilde{z}, \tilde{y}) = Eu(w_0 - \pi_u(k), \tilde{y}). \tag{78}$$

Differentiating the above equality with respect to k yields

$$E\tilde{z}u_1(w_0 + k\tilde{z}, \tilde{y}) = -\pi'_u(k)Eu_1(w_0 - \pi_u(k), \tilde{y}), \tag{79}$$

and so $\pi'_u(0) = 0$ because $E\tilde{z} = 0$. Differentiating once again with respect to k yields

$$E\tilde{z}^2 u_{11}(w_0 + k\tilde{z}, \tilde{y}) = [\pi'^2_u Eu_{11}(w_0 - \pi_u(k), \tilde{y}) - \pi''_u(k)Eu_1(w_0 - \pi_u(k), \tilde{y}). \tag{80}$$

This implies that

$$\pi''_u(0) = -\frac{Eu_{11}(w_0, \tilde{y})}{Eu_1(w_0, \tilde{y})}E\tilde{z}^2. \tag{81}$$

Similarly, for v we have

$$\pi''_v(0) = -\frac{Ev_{11}(w_0, \tilde{y})}{Ev_1(w_0, \tilde{y})}E\tilde{z}^2. \tag{82}$$

Now π_u and π_v can be written in the form of Taylor expansions around $k = 0$:

$$\pi_u(k) = -\frac{Eu_{11}(w_0, \tilde{y})}{Eu_1(w_0, \tilde{y})}E\tilde{z}^2 k^2 + o(k^2) \tag{83}$$

and

$$\pi_v(k) = -\frac{Ev_{11}(w_0, \tilde{y})}{Ev_1(w_0, \tilde{y})}E\tilde{z}^2 k^2 + o(k^2). \tag{84}$$

Then, from (77) we know that, when $k \to 0$, we get $\pi_u < \pi_v$ for $F(x, y)$ such that $F_Y(y)$ has positive support on interval $[n'_1, n'_2]$. This is a contradiction. Q.E.D.

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