An Alternative Representation of the C-CAPM with Higher-Order Risks*

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Abstract

This paper exploits the concept of expectation dependence to propose an alternative representation of the consumption-based capital asset pricing model (C-CAPM). While the first-degree expectation dependence (FED) drives the C-CAPM's riskiness for a risk-averse investor, the second-degree expectation dependence (SED) is required to account for the downside risk faced by a prudent investor. Theoretical and empirical assessments reveal that the expectation dependence-based C-CAPM can realistically match equity and variance risk premia. The consumption SED risk emerges as a fundamental source of uncertainty driving asset prices.

Keywords: C-CAPM, Expectation dependence, Higher-order risk, equity risk premium, variance risk premium

JEL classification: D51, D80, G12

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1. Introduction

The consumption-based capital asset pricing model (C-CAPM), developed by Rubinstein (1976), Lucas (1978), and Breeden (1979) relates an asset's risk premium to the covariance between the return on this asset and an investor's intertemporal marginal rate of substitution. In this pricing framework, the quantity of risk is captured by the covariance of the marginal utility of consumption with the asset return. Assuming a special case of power utility function and joint normal distribution of observable covariates, this quantity of risk can be assessed by the covariance of the asset return with aggregate consumption growth. However, power utility does not combine well with non-normal distributions whereas there is strong evidence that financial returns are non-normally distributed. This underscores the importance of accurately assessing higher-order risks in the determination of asset prices and risk premia.

This study proposes a new representation of the C-CAPM pricing rule based on the concept of expectation dependence that delivers an alternative measure of downside risk. While the riskiness in the C-CAPM is driven by the covariance between the asset return and the marginal utility of consumption, we typically do not have empirical access to the marginal utility of consumption. Instead, consumption and asset return are empirically accessible. A key insight of our contribution is that the covariance between an investor's marginal utility of consumption and an asset's return can be re-expressed in terms of the expectation dependence between empirically accessible covariates, that is, asset return and aggregate consumption growth. The proposed expectation dependence-based C-CAPM highlights higher-order risk quantities and corresponding higher-order risk attitudes without imposing restrictive assumptions on the functional forms of distributions and preferences (Eeckhoudt and Schlesinger 2006, Denuit and Eeckhoudt 2010). Our theoretical representation of the pricing rule determines the optimal asset risk premium as a function of a relative risk aversion index for the first-degree expectation dependence (FED) and a relative prudence index for the second-degree expectation dependence (SED).

The adopted approach is related to the literature on the measurement of risk, which is central to asset pricing. Closely related to our work are papers by Denuit et al. (2015) and Wright (1987) who explore the theoretical foundations of expectation dependence in decision theory. Li (2011) generalizes the concept of expectation dependence to higher orders to characterize how background risks shape the demand for risky assets. Higher risk attitudes are mapped to higher-oder risk quantities in Crainich and Eeckhoudt (2008), Denuit and Eeckhoudt (2010), Eeckhoudt and Schlesinger (2006), and Gollier et al. (2013), among others. Denuit and Scaillet (2004) and Zhu et al. (2016) develop inference frameworks and robust empirical tests for expectation dependence. Another strand of the asset pricing literature builds on the seminal work of Kraus and Litzenberger (1976) and subsequent contributions by Harvey and Siddique (2000), Dittmar (2002), Ang et al. (2006), Martellini and Ziermann (2010), and Lambert and Huebner (2013) to document empirical evidence of risk premia associated with higher-order moments (coskewness and cokurtosis) of portfolio return distributions. Recent asset pricing models with higher moments in consumption growth, driven by left-skewed and fat-tailed shocks, yield realistic premia as argued in Martin (2013), Constantinides and Gosh (2017), and Pohl et al. (2018). By specifying consumption and dividend growth processes in a bad uncertainty-good uncertainty environment, Feunou et al. (2018) extend Bollerslev et al.'s (2009) model and document empirically sound implications such as countercyclical market compensations of risks and sizeable variance risk premia.

This paper focuses on expectation dependence because a more restrictive measure of dependence is not needed to obtain the results. While first-degree expectation dependence and covariance are linked, there is not necessarily a one-to-one correspondence in the signs of these two measures of dependence. For a risk-averse agent, a necessary and sufficient condition for a positive risk premium is positive first-degree expectation dependence. Because positive covariance does not necessarily imply positive first-degree expectation dependence, by corollary, positive (negative) covariance is not sufficient to ensure a positive (negative) risk premium for all return distributions.

Further, first-degree expectation dependence may not be sufficient to set the equilibrium price of an asset. A prudent investor who saves for precautionary motives (Kimball 1990) also cares about the downside risk (Menezes et al. 1980) or a related measure, the second-degree expectation dependence. Lettau et al. (2014) argue that a downside risk-based asset pricing model can successfully price the cross section of various asset classes including equities, equity index options, commodities, sovereign bonds, and currencies. We show for a prudent investor that a positive second-degree expectation dependence commands a higher premium through the aversion to downside risk. Accounting for prudence and related SED risk quantity may help better assess the premium implied by a given risk aversion level.

Using both simulations and international markets data from Campbell (2003), we implement the same estimation procedure to appraise the empirical performance of the expectation dependence representation of the C-CAPM in matching observed equity and variance risk premia. Our SED representation of the C-CAPM yields realistic empirical indexes of risk aversion for equity and variance risk premia.

The paper proceeds as follows. Section 2 presents an alternative expectation dependence-based representation of the pricing rule for a risk-averse representative investor. Section 3 extends the analysis to a prudent investor. Section 4 compares theoretically and via simulations the standard pricing expression with the expectation dependence-based formula of the C-CAPM. Section 5 shows empirically how the expectation dependence-based representation can help improve the empirical performance of the C-CAPM in matching the equity risk premium on international markets and the variance risk premium in the U.S. market. Section 6 concludes. The Appendix contains additional results.

2. C-CAPM for a Risk-Averse Representative Agent

Suppose an investor can freely buy or sell an asset with random payoff $\tilde{x}_{t+1}$ at a price p_t. The asset's gross return is $1 + \tilde{R}_{t+1} = \tilde{x}_{t+1}/p_t$. The investor's preference is represented by a utility function u, with $u' > 0$, $u'' < 0$, and derivatives existing to all orders. Denote the time discount factor by β, and the current consumption by c_t. The Euler equation from the investor's utility maximization problem gives the well-known C-CAPM rule in return form

$$E_t\tilde{R}_{t+1} - R_t^f = -\frac{\mathrm{cov}_t[u'(\tilde{c}_{t+1}), \tilde{R}_{t+1}]}{E_t u'(\tilde{c}_{t+1})}, \tag{1}$$

where E_t is the conditional expectation operator, $\tilde{c}_{t+1}$ is the time $t+1$ consumption, $1+R_t^f = (\beta E_t u'(\tilde{c}_{t+1})/u'(c_t))^{-1}$ is the gross return of the risk-free asset, u' is the marginal utility function, and $E_t\tilde{R}_{t+1} - R_t^f$ is the asset's conditional equity risk premium.

A key challenge in the empirical literature is to reexpress the pricing formulas using the dependence between observed variables, namely, consumption (not the marginal utility of consumption) and the asset's return. To circumvent this challenge, restrictions are commonly imposed on utility functions and distributions.[1] By rewriting the covariance in (1) in terms of expectation dependence measures between observed covariates, we propose an alternative representation of the C-CAPM pricing rule.

2.1. Pricing with First-Order Expectation Dependence

The concept of first-degree expectation dependence (FED) is a stronger definition than correlation (Wright 1987). Formally, consider two continuous random variables $(\tilde{x}, \tilde{y}) \in [\underline{x}, \bar{x}] \times [\underline{y}, \bar{y}]$. Let $F(x, y)$ denote the joint cumulative distribution, and $F_{\tilde{x}}(x)$ and $F_{\tilde{y}}(y)$ be the marginal distributions of $\tilde{x}$ and $\tilde{y}$.

Definition 2.1 *If* $\mathrm{FED}(\tilde{x}|y) = [E\tilde{x} - E(\tilde{x}|\tilde{y} \leq y)] \geq 0$ *for all* $y \in \mathbb{R}$*, then* $\tilde{x}$ *is positive first-degree expectation dependent on* $\tilde{y}$*.*

Wright (1987, p. 113) interprets positive FED as follows: "When we discover $\tilde{y}$ is small, in the precise sense that we are given the truncation $\tilde{y} \leq y$, our expectation of $\tilde{x}$ is revised downward." Here, we exploit a useful equivalence between the sign of $\mathrm{cov}\left(f\left(\tilde{x}\right), \tilde{y}\right)$ and positive (or negative) FED for any monotone function f (Theorem 3.1 in Wright 1987).[2]

The consumption growth between time t and $t+1$ is $\tilde{g}_{t+1} = \tilde{c}_{t+1}/c_t$ and takes values in $[\underline{g}, \bar{g}]$. The second derivative of the utility function is negative ($u'' < 0$) because a risk-averse investor's marginal utility is monotonically decreasing. Using integration by parts as in Cuadras (2002) and results in Tesfatsion (1976) and Wright (1987), the covariance in the pricing equality can be reexpressed in terms of the FED and a second-order preference index:

$$\mathrm{cov}_t[u'(c_t\tilde{g}_{t+1}), \tilde{R}_{t+1}] \quad = \quad \int_{\underline{g}}^{\bar{g}} \mathrm{FED}(\tilde{R}_{t+1}|g_{t+1}) c_t u''(c_t g_{t+1}) F_{\tilde{g}_{t+1}}(g_{t+1}) dg_{t+1}. \qquad (2)$$

Appendix A1 gives the proof of (2). Define $\mathrm{RR}_2(c_{t+1}) = -c_{t+1}u''(c_{t+1})/u'(c_{t+1})$ as the Arrow-Pratt relative risk aversion index, $\mathrm{MRS}_{c_{t+1},c_t} = u'(c_{t+1})/u'(c_t)$ as the intertemporal marginal rate of substitution, and set $[\beta(1+R_t^f)]^{-1} = E_t u'(\tilde{c}_{t+1})/u'(c_t)$. We use (2) to rewrite (1) as

$$\frac{E_t\tilde{R}_{t+1} - R_t^f}{\beta(1+R_t^f)} = \int_{\underline{g}}^{\bar{g}} \underbrace{\mathrm{FED}(\tilde{R}_{t+1}|g_{t+1})F_{\tilde{g}_{t+1}}(g_{t+1})}_{\text{FED risk}}\,\mathrm{RR}_2(c_t g_{t+1})\,\mathrm{MRS}_{c_t g_{t+1},c_t}\,\frac{1}{g_{t+1}}dg_{t+1}. \qquad (3)$$

The quantity of consumption risk in (3) is measured by the FED of asset return with consumption growth. The other terms reflect the price of risk for a risk-averse investor. An asset commands a positive equity risk premium if and only if its return is positively first-degree expectation dependent with consumption growth.[3] Intuitively in the FED representation of the C-CAPM, an asset bears a positive risk premium when its expected return is lowered as we observe consumption growth $\tilde{g}$ below a reference level g (Wright 1987).[4] Conversely, an asset's equity risk premium is negative if and only if its return is negatively first-degree expectation dependent with consumption growth. In this case, the asset's expected return is revised up as consumption growth falls below a reference level, thus providing a digital option hedge against a shortfall of $\tilde{g}$ below g (Denuit et al. 2015).

We stress that what matters for the C-CAPM pricing is the covariance between the marginal utility of consumption $u'(\tilde{c})$ and the asset's return $\tilde{R}$. However, we typically do not have empirical access to $u'(\tilde{c})$. Instead, $\tilde{c}$ is empirically accessible along with $\tilde{R}$. We show in (3) that the FED between asset return and consumption growth (which are both empirically accessible) determines the asset's riskiness for flexible distributions and preferences. Because FED is a stronger dependence measure than correlation, a positive (negative) covariance between asset return and consumption growth is a necessary but not sufficient condition for a risk-averse agent demanding a positive (negative) equity risk premium. The FED pricing formula highlights the dependence of asset return with consumption growth rather than with the marginal utility of consumption which is not readily observable from available data.

3. C-CAPM for a Risk-Averse and Prudent Representative Agent

Now, consider a representative agent that is not only risk-averse ($u'' < 0$) but also prudent ($u''' > 0$), where u''' denotes the third derivative of the utility function. The concept of prudence and its relationship to precautionary savings (Kimball 1990) is a well-established behavior in the risk literature (Gollier 2001, Keenan and Snow 2002). Gollier et al. (2013) provide a careful review of analytical developments on prudence. All prudent agents dislike any increase in downside risk in the sense of Menezes et al. (1980), as argued by Chiu (2005) and Crainich and Eeckhoudt (2008). This section discusses expectation dependence conditions for asset prices and equity premia when the agent is risk-averse and prudent.

3.1. Pricing with Second-Order Expectation Dependence

To characterize the impact of prudence in the expectation dependence representation of the C-CAPM, we can integrate the right-hand term of (2) by parts. We get

$$\mathrm{cov}_t[u'(c_t\tilde{g}_{t+1}), \tilde{R}_{t+1}] = c_t u''(c_t\bar{g})\,\mathrm{cov}_t(\tilde{R}_{t+1}, \tilde{g}_{t+1}) - \int_{\underline{g}}^{\bar{g}} \mathrm{SED}(\tilde{R}_{t+1}|g_{t+1}) c_t^2 u'''(c_t g_{t+1}) dg_{t+1}, \quad (4)$$

where SED is the second-order expectation dependence between asset return and consumption growth, a concept formally discussed in Li (2011). See Appendix A1 for the proof.[5]

Definition 3.1 *If* $\mathrm{SED}(\tilde{x}|y) = \int_{\underline{y}}^{y} [E\tilde{x} - E(\tilde{x}|\tilde{y} \le t)] F_{\tilde{y}}(t) dt = \int_{\underline{y}}^{y} \mathrm{FED}(\tilde{x}|t) F_{\tilde{y}}(t) dt \ge 0$ *for* *all y, then $\tilde{x}$ is positive second-degree expectation dependent on $\tilde{y}$.*

The pricing formula in (3) can then be restated as

$$\frac{E_t\tilde{R}_{t+1} - R_t^f}{\beta(1+R_t^f)} = \underbrace{\mathrm{cov}_t(\tilde{R}_{t+1}, \tilde{g}_{t+1})}_{\text{covariance risk}} \mathrm{RR}_2(c_t\bar{g})\,\mathrm{MRS}_{c_t\bar{g},c_t}\,\frac{1}{\bar{g}} + \int_{\underline{g}}^{\bar{g}} \underbrace{\mathrm{SED}(\tilde{R}_{t+1}|g_{t+1})}_{\text{SED risk}} \mathrm{RR}_3(c_t g_{t+1})\,\mathrm{MRS}_{c_t g_{t+1},c_t}\,\frac{1}{g_{t+1}^2} dg_{t+1}, \tag{5}$$

where $\mathrm{RR}_3(c_{t+1}) = c_{t+1}^2 u'''(c_{t+1})/u'(c_{t+1})$ defines a relative downside risk aversion index as in Modica and Scarsini (2005).[6] The representation in (5) suggests that $\mathrm{cov}_t(\tilde{R}_{t+1}, \tilde{g}_{t+1}) > 0$ is only a necessary condition for all non-satiable, risk-averse and prudent agents to ask for a positive risk premium. Using only the covariance of asset return and consumption growth introduces a distortion in the pricing formula when the third derivative of the utility function is different from zero—or when the utility function is not quadratic. The last term on the right-hand side of (5) reflects how the SED risk affects the asset price through the intensity of downside risk aversion (Hogan and Warren 1974, Bawa and Lindenberg 1977, Price et al. 1982). Appendix A3 presents the pricing equations in (3) and (5) in price form.

To further provide an intuitive link between SED and downside risk, we follow Denuit et al. (2015) who establish that the k^{th} order expectation dependence can be reexpressed as

$$k^{th}ED(\tilde{R}|g) = -\frac{1}{(k-1)!}\,\mathrm{cov}((g-\tilde{g})_+^{k-1}, \tilde{R}), \tag{6}$$

where $k \geq 2$, ! denotes the factorial function, and $(\bullet)_+$ is equal to the positive part of its argument. For $k = 2$, the equivalence in (6) clearly shows that $\mathrm{SED}(\tilde{R}|g)$ is minus the covariance between the asset's return $\tilde{R}$ and the payoff of a put option written on consumption growth $\tilde{g}$, protecting against its shortfall below a reference level (or strike price) g. Thus, SED$(\tilde{R}|g)$ is related to the second-order lower partial cross-moment of $\tilde{R}$ and $\tilde{g}$, and can be intuitively interpreted as a measure of downside risk computed below a benchmark level of consumption growth g. As the benchmark level g grows to infinity, SED$(\tilde{R}|g \to +\infty)$ approaches $\text{cov}(\tilde{R}, \tilde{g})$. This equivalence is empirically appealing, as it facilitates the computation of SED with actual data.

Thus, a positive integrated consumption SED in (5) is obtained when there are more positive lower partial covariances between asset return and consumption growth, that is, when stock market portfolio returns are more positively tied to consumption growth in the left part of the joint distribution. A positive SED reinforces the effect of a positive covariance between asset return and consumption growth to obtain a positive risk premium. In that case, the stock market portfolio does not offer a hedge against the downside consumption risk, and a prudent representative investor demands a higher premium for bearing that downside risk. Note that for simplicity, we label the pricing expression in (5) as the SED C-CAPM representation, even though this equation also involves a covariance term.

The expectation dependence-based pricing formulas in (3) and (5) can also be stated in terms of absolute rather than relative risk index—see Appendix A4. However, while an absolute risk index is theoretically useful to make some qualitative statements, it might be challenging to assess with actual data. Specifically, assuming a constant absolute risk aversion utility function is not empirically grounded, as it entails an increasing relative risk aversion—absolute risk aversion times consumption—with a growing economy. An agent with increasing relative risk aversion will invest less in the risky asset as he becomes richer, which is quite nonintuitive with respect to the large empirical evidence suggesting otherwise.

Using data on portfolio holdings of U.S. households, Bucciol and Miniaci (2011) document that the rich tend to invest more of their relative wealth in stocks.

4. Comparison of Alternative Representations of the C-CAPM

In this section, we compare the standard (STD) formula of the C-CAPM in (1) with the first- and the second-order expectation dependence representations of the pricing rule in (3) and (5). To this end, we compute an n^{th} order Taylor expansion of the alternative representations discussed above—see Appendix A5 for derivations. We stress that all three (STD, FED, SED) representations of the C-CAPM are theoretically equivalent in the limit ($n \to \infty$). For a given finite order Taylor approximation, we compare these alternative representations of the C-CAPM analytically and empirically using simulations. This comparison is intended to assess potential differences in the measurements of various risk quantities associated with higher risk attitudes in the alternative formulations.

4.1. Analytical Approximation

Following Modica and Scarsini (2005), we define the k^{th} order relative risk index as $\mathrm{RR}_k(c) = c^{k-1}(-1)^{k-1}u^{(k)}/u'$, with $u^{(k)}$ denoting the k^{th} derivative of the utility function and $k \geq 2$. Note that for an investor equipped with a power utility function $u(c) = \frac{c^{1-\gamma}-1}{1-\gamma}$, the $(i+1)^{th}$ order relative risk index is given by $\mathrm{RR}_{i+1}(c) = \prod_{j=1}^{i}(\gamma + j - 1)$, with $i \geq 1$. Moreover, the conditional expectation of the intertemporal marginal rate of substitution $E_t MRS_{\tilde{c}_{t+1},c_t} = [\beta(1+R^f)]^{-1}$ becomes $E_t \tilde{g}_{t+1}^{-\gamma}$.

It is worth stressing that the alternative representations of the C-CAPM and related

Taylor approximations do not only hold for constant relative risk aversion preferences. While the analytical formulas are stated in terms of relative risk aversion indexes, the pricing expressions and Taylor approximations can be rewritten in terms of absolute risk aversion indexes, as discussed in Appendix A4. In that case, the k^{th} order absolute risk intensity is measured by $\mathrm{AR}_k(c) = (-1)^{k-1}u^{(k)}/u'$, as in Crainich and Eeckhoudt (2008).[7]

4.1.1. STD Representation

For the analytical assessment, we follow Dittmar (2002) and perform a Taylor expansion at the order n of the standard representation of the C-CAPM in (1). The Taylor approximation details are presented in Appendix A5. We have

$$\begin{aligned} \frac{E_t\tilde{R}_{t+1} - R_t^f}{\beta(1+R_t^f)} &= -\frac{1}{u'(c_t)}\,\mathrm{cov}_t(u'(c_t\tilde{g}_{t+1}), \tilde{R}_{t+1}), \\ &= \sum_{i=1}^{n} \frac{(-1)^{i+1}}{i!}\,\mathrm{RR}_{i+1}(c_t)\,\mathrm{cov}_t((\tilde{g}_{t+1}-1)^i, \tilde{R}_{t+1}) + o(g^n). \end{aligned} \tag{7}$$

In this representation, the first derivative of the utility function is approximated.

4.1.2. FED Representation

We now turn to the FED representation of the C-CAPM in (3) that features the second derivative of the utility function. Here, we compute n^{th} order Taylor polynomials of u'' (not u' as in the standard representation). This expansion, detailed in Appendix A5, yields

$$\begin{aligned} \frac{E_t\tilde{R}_{t+1} - R_t^f}{\beta(1+R_t^f)} &= \int_{\underline{g}}^{\bar{g}} \mathrm{FED}(\tilde{R}_{t+1}|g_{t+1})F_{\tilde{g}_{t+1}}(g_{t+1})[-c_t\frac{u''(c_t g_{t+1})}{u'(c_t)}]dg_{t+1} \\ &= \sum_{i=1}^{n+1} \frac{(-1)^{i+1}}{i!}\,\mathrm{RR}_{i+1}(c_t)\,\mathrm{cov}((\tilde{g}_{t+1}-1)^i, \tilde{R}_{t+1}) + o(g^{n+1}). \end{aligned} \tag{8}$$

We observe that, for the same expansion order n, the FED representation adds an extra $(n+2)^{th}$ degree of risk $\frac{(-1)^{n+2}}{(n+1)!}\,\mathrm{RR}_{n+2}(c_t)\,\mathrm{cov}((\tilde{g}_{t+1}-1)^{n+1},\tilde{R}_{t+1})$.

4.1.3. SED Representation

We also expand the SED representation of the C-CAPM in (5) at the order n using Taylor series. By exploiting the equivalence in (6), we get

$$\begin{aligned}
\frac{E_t\tilde{R}_{t+1}-R_t^f}{\beta(1+R_t^f)} &= \mathrm{cov}_t(\tilde{g}_{t+1},\tilde{R}_{t+1})(-c_t\frac{u''(c_t\bar{g})}{u'(c_t)})+\int_{\underline{g}}^{\bar{g}}\mathrm{SED}(\tilde{R}_{t+1}|g_{t+1})(-c_t^2\frac{u'''(c_tg_{t+1})}{u'(c_t)})dg_{t+1} \\
&= \mathrm{cov}_t(\tilde{g}_{t+1},\tilde{R}_{t+1})[\sum_{i=1}^{n+1}\frac{(-1)^{i+1}}{(i-1)!}\,\mathrm{RR}_{i+1}(c_t)(\bar{g}-1)^{i-1}] \\
&\quad+\int_{\underline{g}}^{\bar{g}}\mathrm{cov}_t(-(g_{t+1}-\tilde{g}_{t+1})_+,\tilde{R}_{t+1})[\sum_{i=1}^{n+1}\frac{(-1)^{i+2}}{(i-1)!}\,\mathrm{RR}_{i+2}(c_t)(g_{t+1}-1)^{i-1}]dg_{t+1} \\
&\quad+o(g^{n+2}). \qquad (9)
\end{aligned}$$

Appendix A5 contains step-by-step derivations. For the same order of approximation n, the SED representation includes an additional effect of the $(n+3)^{th}$ degree of risk aversion $\frac{(-1)^{n+3}}{n!}\,\mathrm{RR}_{n+3}(c_t)$. Higher-order risk quantities are also measured differently in the SED representation compared to the STD and FED approaches. Namely, risk quantities in the SED representation (9) involve $\int_{\underline{g}}^{\bar{g}}\mathrm{cov}_t(-(g_{t+1}-\tilde{g}_{t+1})_+,\tilde{R}_{t+1})(g_{t+1}-1)^i dg_{t+1}$, for $i=1,\cdots,n$. This integrated i^{th} power weighted SED of asset return on consumption growth captures downside risk patterns through the negative covariance between the asset's return and the payoff of a put option written on consumption growth, scaled by higher powers of consumption growth. The empirical assessment below shows that, for a given order of approximation, the SED representation outperforms the STD and FED representations of the C-CAPM.

4.1.4. Illustrative example: third-order Taylor approximation

To provide a more intuitive illustration of the alternative representations, we present the approximations of the pricing formulas using a third-order Taylor expansion. Following Dittmar (2002), among others, we truncate the Taylor series expansion after the third order to illustrate intuitively the role of higher-order risk preferences—beyond the standard risk aversion—and related higher-order risk quantities in asset pricing.

By setting $n = 3$ for the approximated STD representation in (7), we obtain

$$\begin{aligned}\frac{E_t\tilde{R}_{t+1} - R_t^f}{\beta(1 + R_t^f)} &= \mathrm{RR}_2(c_t)\,\mathrm{cov}_t(\tilde{g}_{t+1}, \tilde{R}_{t+1}) - \frac{1}{2}\,\mathrm{RR}_3(c_t)\,\mathrm{cov}_t((\tilde{g}_{t+1} - 1)^2, \tilde{R}_{t+1}) \\ &\quad + \frac{1}{6}\,\mathrm{RR}_4(c_t)\,\mathrm{cov}_t((\tilde{g}_{t+1} - 1)^3, \tilde{R}_{t+1}) + o(g^3), \qquad (10)\end{aligned}$$

which highlights, beyond the second-order risk aversion, the role of prudence (RR_3) and temperance (RR_4) in the valuation of a risky asset. Eeckhoudt and Schlesinger (2006) rely on behavioral characteristics of risk apportionment—preference for disaggregating the harms—to provide intuitive definitions of these higher-order risk attitudes. In Eeckhoudt and Schlesinger's (2006) framework, harms represent detrimental changes to wealth. Using simple lottery pairs, the authors show that a prudent investor with a given wealth, facing two distinct harms (a sure loss and a zero-mean risk), prefers to apportion them by placing one in each state. This echoes, Kimball's (1990) interpretation of prudence as the propensity to prepare and forearm oneself in the face of uncertainty or as the intensity of precautionary saving motives. Similarly, a temperate investor with a given wealth, facing two distinct harms (two independent zero-mean risks), prefers attaching one in each state rather than receiving both in one state. The quantities of risk related to prudence and temperance in the standard representation of C-CAPM are captured by the coskewness $(\mathrm{cov}_t((\tilde{g}_{t+1}-1)^2, \tilde{R}_{t+1}))$ and the cokurtosis $(\mathrm{cov}_t((\tilde{g}_{t+1}-1)^3, \tilde{R}_{t+1}))$ of the consumption growth with the asset's return.

For $n = 3$, the approximate FED pricing relation in (8) writes

$$\begin{aligned}\frac{E_t\tilde{R}_{t+1} - R_t^f}{\beta(1+R_t^f)} &= \mathrm{RR}_2(c_t)\,\mathrm{cov}_t(\tilde{g}_{t+1}, \tilde{R}_{t+1}) - \frac{1}{2}\,\mathrm{RR}_3(c_t)\,\mathrm{cov}_t((\tilde{g}_{t+1}-1)^2, \tilde{R}_{t+1}) \\ &\quad + \frac{1}{6}\,\mathrm{RR}_4(c_t)\,\mathrm{cov}_t((\tilde{g}_{t+1}-1)^3, \tilde{R}_{t+1}) \\ &\quad - \frac{1}{24}\,\mathrm{RR}_5(c_t)\,\mathrm{cov}_t((\tilde{g}_{t+1}-1)^4, \tilde{R}_{t+1}) + o(g^4). \qquad (11)\end{aligned}$$

This shows the contribution of the fifth-order preference for risk apportionment (RR_5), also known as edginess or reactivity to multiple risks from precautionary motives (Lajeri-Chaherli 2004). The associated risk quantity in the FED approach is measured by the hyper coskewness $(\mathrm{cov}_t((\tilde{g}_{t+1}-1)^4, \tilde{R}_{t+1}))$ of the consumption growth and asset return.

When $n = 3$, the SED representation in (9) implies

$$\begin{aligned}\frac{E_t\tilde{R}_{t+1} - R_t^f}{\beta(1+R_t^f)} &= \mathrm{RR}_2(c_t)\,\mathrm{cov}_t(\tilde{g}_{t+1}, \tilde{R}_{t+1}) \\ &\quad - \mathrm{RR}_3(c_t)\left[\int_{\underline{g}}^{\bar{g}} \mathrm{cov}_t(-(g_{t+1}-\tilde{g}_{t+1})_+, \tilde{R}_{t+1})dg_{t+1} + (\bar{g}-1)\,\mathrm{cov}_t(\tilde{g}_{t+1}, \tilde{R}_{t+1})\right] \\ &\quad + \mathrm{RR}_4(c_t)\left[\int_{\underline{g}}^{\bar{g}} \mathrm{cov}_t(-(g_{t+1}-\tilde{g}_{t+1})_+, \tilde{R}_{t+1})(g_{t+1}-1)dg_{t+1} + \frac{1}{2}(\bar{g}-1)^2\,\mathrm{cov}_t(\tilde{g}_{t+1}, \tilde{R}_{t+1})\right] \\ &\quad - \mathrm{RR}_5(c_t)\left[\frac{1}{2}\int_{\underline{g}}^{\bar{g}} \mathrm{cov}_t(-(g_{t+1}-\tilde{g}_{t+1})_+, \tilde{R}_{t+1})(g_{t+1}-1)^2dg_{t+1} + \frac{1}{6}(\bar{g}-1)^3\,\mathrm{cov}_t(\tilde{g}_{t+1}, \tilde{R}_{t+1})\right] \\ &\quad + \frac{1}{6}\,\mathrm{RR}_6(c_t)\int_{\underline{g}}^{\bar{g}} \mathrm{cov}_t(-(g_{t+1}-\tilde{g}_{t+1})_+, \tilde{R}_{t+1})(g_{t+1}-1)^3dg_{t+1} + o(g^5), \qquad (12)\end{aligned}$$

which shows that the sixth-order risk attitude (RR_6) is also relevant for the pricing of a risky asset (Deck and Schlesinger 2014). In addition, we notice for the same order of approximation that risk quantities are measured differently in the SED representation as compared to the STD and FED representations. Namely, risk quantities in the SED representation of the pricing rule (12) involve $\int_{\underline{g}}^{\bar{g}} \mathrm{cov}_t(-(g_{t+1} - \tilde{g}_{t+1})_+, \tilde{R}_{t+1})(g_{t+1} - 1)^n dg_{t+1}$. This integrated n^{th} power weighted SED of asset return on consumption growth captures finer information about downside risk patterns and helps improve the empirical performance of the C-CAPM.

4.2. Monte Carlo Experiments

We run various simulations to assess the empirical performance of the alternative representations (STD, FED, SED) of the C-CAPM. Specifically, we simulate the joint distribution of consumption growth and return for a wide range of dependence structures.

4.2.1. Modelling the Joint Distribution of Return and Consumption Growth

We start from

$$\begin{aligned} \tilde{R}_{t+1} &= E(\tilde{R}) + \phi_{\tilde{R}}(\tilde{R}_t - E(\tilde{R})) + \sqrt{(1 - \phi_{\tilde{R}}^2)}\sigma(\tilde{R})\tilde{R}_{t+1}^*, \\ \tilde{g}_{t+1} &= E(\tilde{g}) + \phi_{\tilde{g}}(\tilde{g}_t - E(\tilde{g})) + \sqrt{(1 - \phi_{\tilde{g}}^2)}\sigma(\tilde{g})\tilde{g}_{t+1}^*, \end{aligned} \quad (13)$$

where $\tilde{R}^*$ and $\tilde{g}^*$ are demeaned and standardized return and consumption growth variables, referred to as standardized innovations. To form the joint distribution of asset return and consumption growth we use a copula (Sklar 1959) to assemble the univariate marginal distributions. The joint density is cast as

$$f_{\tilde{R},\tilde{g}}(R, g) = f_{\tilde{R}}(R) \times f_{\tilde{g}}(g) \times \mathbf{c}(F_{\tilde{R}}(R), F_{\tilde{g}}(g)), \quad (14)$$

where $\mathbf{c}$ is the density of a copula $\mathbf{C}$ characterizing the dependence structure of the bivariate distribution. The density of the copula is computed as $\mathbf{c}(v_1, v_2) = \frac{\partial^2 \mathbf{C}(v_1,v_2)}{\partial v_1 \partial v_2}$, where $(v_1, v_2) \in [0,1]^2$. Moreover, $f_{\tilde{R}}(R)$ and $f_{\tilde{g}}(g)$ denote the marginal densities of return and consumption growth, while $F_{\tilde{R}}(R)$ and $F_{\tilde{g}}(g)$ are the corresponding marginal cumulative distribution functions.

4.2.2. Modelling the Marginal Distributions

We model the marginal distributions of return and consumption growth, by fitting observed U.S. quarterly series (1947.2-2011.3). A complete description of the data set is provided in Section 5. We employ Hansen's (1994) skew t distribution, a generalization of the Student's t law that has been shown to deliver a good fit for marginal distributions in several financial applications (Jondeau and Rockinger 2003, Patton 2004). Beyond location and scale, the skew t distribution allows for a parsimonious and flexible specification of asymmetry as well as fat-tailedness.[8] The skew t distribution is characterized by a set of two parameters. The first (degree of freedoom) parameter $2 < \nu < \infty$ controls the thickness of the tails, while the second parameter $-1 < \lambda < 1$ drives the level of asymmetry in the distribution. For $\lambda = 0$, this distribution collapses to the (symmetric) Student's t. With other specific parameter values, well-known nested distributions such as skew Gaussian ($\nu \to \infty$) and standard Gaussian ($\nu \to \infty, \lambda = 0$) are obtained. We refer the reader to Hansen (1994) for further details on the skew t distribution.

Table 1 reports the summary statistics for return and consumption growth series which display negative skweness and excess kurtosis. This observation is further supported by the (maximum likelihood) parameter estimates of a skew t distribution fitted on these series, as displayed in the last four columns of Table 1.

The top panel in Figure 1 shows that the skew t density offers a good fit to the empirical histogram. The bottom panel reveals that the parametric specification of the marginal distributions fits well the tails of each series. Next, we turn to modeling the dependence structure between return and consumption growth series.

4.2.3. Modeling the Dependence Structure with the Symmetrized Joe-Clayton Copula

To provide a rich description of the dependence structure between return and consumption growth, we consider the quantile dependence that measures the strength of association in the joint lower and upper tails of any bivariate distribution. A quantile dependence at a level $q \in (0,1)$ is defined as

$$\tau(q) = \begin{cases} \tau_D(q) = \mathbb{P}\left(F_{\tilde{R}}\left(R\right) \leq q | F_{\tilde{g}}\left(g\right) \leq q\right) = \frac{\mathbf{C}(q,q)}{q}, & if\ 0 < q \leq 0.5 \\ \tau_U(q) = \mathbb{P}\left(F_{\tilde{R}}\left(R\right) > q | F_{\tilde{g}}\left(g\right) > q\right) = \frac{1-2q+\mathbf{C}(q,q)}{1-q}, & if\ 0.5 < q < 1. \end{cases} \quad (15)$$

Thus, instead of summarizing the association between two covariates by a single number such as the linear correlation, the quantile dependence offers a finer information on the dependence structure of the joint distribution. This measure can reveal potential dependence asymmetries as we move from the middle of the distribution ($q = 0.5$) towards the left tail ($q < 0.5$) or the right tail ($q > 0.5$).[9]

We choose the symmetrized Joe-Clayton (SJC) copula ($\mathbf{C}_{SJC}$) discussed in Patton (2006) to model the dependence between return and consumption growth series.[10] This copula allows to specify a wide range of dependence patterns in the joint distribution. Following

Patton (2006), we parameterize the SJC copula using the coefficient of upper tail dependence

$$\tau_U = \lim_{q \to 1} \tau_U(q), \tag{16}$$

and lower tail dependence

$$\tau_D = \lim_{q \to 0} \tau_D(q), \tag{17}$$

where $(\tau_U, \tau_D) \in (0,1)^2$. Thus, tail dependence parameters of the SJC copula are simply quantile dependence measures computed in the limits (both ends) of the support of the bivariate distribution. Intuitively, $\tau_D > \tau_U$ (resp. $\tau_U > \tau_D$) induces a higher probability of the two covariates taking extreme values in the lower (resp. upper) than in the upper (resp. lower) quadrant of their joint distribution, thus reflecting downside risk (resp. upside potential).[11] The estimated tail parameter values for the SJC copula fitted to U.S. quarterly return and consumption growth series (1947.2-2011.3) are $\tau_U = 5.150 \times 10^{-5}$ and $\tau_D = 0.173$ with the corresponding standard errors $SE(\tau_U) = 1.264 \times 10^{-3}$ and $SE(\tau_D) = 0.073$. Clearly, the estimated left tail dependence is stronger than the right tail dependence.

Figure 2 illustrates the joint distribution of return and consumption growth. The top left panel shows the scatter plot of standardized innovations in return and consumption growth, with a linear correlation of 0.265. The top right panel in Figure 2 exhibits the isoprobability contour of the fitted SJC copula with skew t marginals estimated in Table 1. We observe that the copula density contours are more tightly clustered around the lower left than the upper right quadrant. Consistent with the tail parameter estimates, this empirical observation suggests that the strength of dependence conditional on downside movements or "bad" times – driven by a contraction of consumption – is higher than the strength of dependence conditional on upside movements or "good" times – driven by a expansion of consumption. Such an asymmetric dependence suggests that the U.S. market portfolio is exposed to downside consumption risk.[12]

4.2.4. Simulation scheme

Assume that the investor's preference is characterized by a power utility function, and consider that the dependence of the random vector $(\tilde{R}, \tilde{g})$ is fully described by a copula $\mathbf{C}_{SJC}(F_{\tilde{R}}(R), F_{\tilde{g}}(g); \tau_U, \tau_D)$. For a given dependence structure (fixed values of τ_U and τ_D), we implement the following algorithm to draw N_{sim} pairs of return and consumption growth:

Step 1 Generate $(x_{R^*}, x_{g^*})_0^T$ using the copula quantile function $\mathbf{C}_{SJC}^{-1}(F_{\tilde{R}}(R^*), F_{\tilde{g}}(g^*); \tau_U, \tau_D)$;

Step 2 Use the univariate cumulative distribution function $\Phi_{\mathbf{C}}$ associated with the copula to evaluate $v_{R^*} = \Phi_{\mathbf{C}_{SJC}}(x_{R^*})$ and $v_{g^*} = \Phi_{\mathbf{C}_{SJC}}(x_{g^*})$;

Step 3 Compute $R^* = F_{\tilde{R}}^{-1}(v_{R^*})$ and $g^* = F_{\tilde{g}}^{-1}(v_{g^*})$ using the quantile functions of the marginal distributions $F_{\tilde{R}}$ and $F_{\tilde{g}}$ with parameters estimated in Table 1;

Step 4 Build $(R, g)_0^T$ from simulated zero mean standardized pairs $(R^*, g^*)_0^T$ using (13), where $T = 250$ is set close to the sample size of observed quarterly series;

Step 5 Repeat steps 1 to 4 $N_{sim} = 1000$ times;

Step 6 For a given utility function u and a fixed level of risk aversion, calculate the expected excess return $E(eR) = -\operatorname{cov}(u'(cg), R)/E(u'(cg))$ in the population (of size

TN_{sim}) from which samples are drawn; with the power utility function, $E(eR) = -\operatorname{cov}(g^{-\gamma_{sim}}, R)/E(g^{-\gamma_{sim}})$, where γ_{sim} is a chosen value of relative risk aversion;

Step 7 For each simulation in N_{sim}, assume (as in Campbell and Viceira 2002) that the excess return follows an AR(1) process, and construct a time series $eR_{t+1} = E(eR)+\phi_{eR}(eR_t - E(eR))+\sqrt{(1-\phi_{eR}^2)}\sigma(eR)\eta_{t+1}$, where $E(eR)$ is calculated in Step 6, and η_{t+1} has a zero mean standardized skew t ($\nu = 8.149, \lambda = -0.194$) distribution as R^*; the remaining parameters $\sigma(eR) = 0.1$, and $\phi_{eR} = 0.1$ are set close to their empirical values;

Step 8 Calibrate the relative risk aversion from the simulated data; solve for the relative risk aversion parameter by computing a numerical root of the nonlinear pricing functions in the STD, FED, and SED formulas of the C-CAPM in (10-12) with power utility;[13]

4.2.5. Simulation results with different tail dependence parameters

Table 2 contains the simulation results assuming relative risk aversion values of $\gamma_{sim} = 5$ (Panels A, B, C), and $\gamma_{sim} = 10$ (Panels D, E, F). Simulations are performed for various dependence structures between asset return and consumption growth series: (1) stronger left tail dependence ($\tau_U < \tau_D$), (2) stronger right tail dependence ($\tau_U > \tau_D$), and (3) symmetric tail dependence ($\tau_U = \tau_D$). In each panel, tail dependence parameters are chosen to imply the same level of linear correlation for ease of comparison. We consider different simulation scenarios featuring high (Panels A, D), medium (Panels B, E), and low (Panels C, F) linear correlation values between the series. For the simulation simL1 in Panels C and F, tail dependence parameters are set close to the fitted values in the data.

For each set of simulations, we compute the $bias = N_{sim}^{-1}\sum_{i=1}^{N_{sim}}(\gamma_i^{calibrated} - \gamma_{sim})$ and the root-mean-squared error denoted by $rmse = \sqrt{N_{sim}^{-1}\sum_{i=1}^{N_{sim}}(\gamma_i^{calibrated} - \gamma_{sim})^2}$ of the calibrated relative risk aversion coefficients from the alternative (STD, FED, SED) representations of the pricing relation. These are shown in columns 5-10 of Table 2. Looking at the last four columns in Table 2, our simulations reveal that the SED representation delivers more accurate relative risk aversion values across the board and markedly improves upon the FED and the STD representations. For a relative risk aversion $\gamma_{sim} = 10$, we observe on average across our simulations about 50% and 70% reduction in *bias* and *rmse* for the SED approach relative to the STD representation; whereas the improvement in *bias* and *rmse* for the FED approach is about 42% and 60%, respectively. When the relative risk aversion value is fixed at $\gamma_{sim} = 5$, the improvement in accuracy for the SED representation is larger, with 70% and 80% reduction in *bias* and *rmse* compared to the STD representation. Note that $\gamma^{calibrated}$ for the SED representation is upward-biased—though the SED *bias* is lower than FED and STD *biases*. This upward bias is in line with the findings in Kocherlakota (1996) and the generalized method of moments estimates of the relative risk aversion in Martin (2013)'s simulated disaster C-CAPM with higher order cumulants. Focusing on the tail dependence, the simulation results also show that relative risk aversion values for the SED representation tend to be more accurate when the left tail dependence is stronger (simH1, simM1, SimL1) than for symmetric (simH3, simM3, SimL3) and stronger right tail dependence structures (simH2, simM2, SimL2).[14]

4.2.6. Simulation results with different autocorrelation, skewness, and kurtosis parameters

To further assess the performance of the SED representation, we turn to Table 3 which investigates the effects of autocorrelation, skewness, and kurtosis in the distributions of asset return and consumption growth on the calibrated relative risk aversion. This simulation exercise is carried out for a relative risk aversion $\gamma_{sim} = 5$. In the simulation simP1, the chosen parameters imply the following vectors of skewness $(s_R, s_g) = (-0.6, -0.3)$ and kurtosis $(k_R, k_g) = (5.4, 7.3)$.[15] simP2 reports the simulations when there is no autocorrelation, $(\phi_R, \phi_g) = (0, 0)$. Setting the autocorrelation to zero in simP2 entails slightly more negative skewness $(s_R, s_g) = (-0.6, -0.5)$ and higher kurtosis $(k_R, k_g) = (5.6, 12.1)$ than in simP1 as expected. We see that the accuracy of the SED representation improves compared to the standard approach when the return and consumption growth series become more negatively skewed and fat-tailed. The *bias* and *rmse* reduction for the SED representation increases from 72% and 82% in simP1 to 81% and 87% in simP2. When we assume zero skewness $(s_R, s_g) = (0, 0)$ and positive excess kurtosis $(k_R, k_g) = (5.0, 7.0)$ in simP3, the SED approach improves moderately over the standard representation, as the *bias* and *rmse* decline by 10% and 15%. simP4 investigates the case of weakly negative skewness $(s_R, s_g) = (-0.3, -0.1)$ and almost no excess kurtosis $(k_R, k_g) = (3.1, 3.0)$. In simP4, the improvement in accuracy for the SED representation is also relatively modest, with 18% and 26% reduction in *bias* and *rmse* compared to the STD representation. Finally, simP5 assumes no autocorrelation $(\phi_R, \phi_g) = (0, 0)$, no skewness $(s_R, s_g) = (0, 0)$, and almost no excess kurtosis $(k_R, k_g) = (3.0, 3.1)$. We see that the SED approach has a similar precision as the FED and STD representations when the return and consumption growth distributions are nearly Gaussian. Overall, the accuracy improvement in the simulation exercise supports the SED representation of the pricing equation, especially in the presence of downside risk.

5. Empirical Analysis for International Markets

For our empirical analysis, we use the same data set as Campbell (2003) for comparability and implement the same numerical estimation procedure as in the simulation exercise. The data can be downloaded from the author's website.[16] This international data set combines Morgan Stanley Capital International stock market data with macroeconomic data on consumption, interest rates, and the price index from the International Financial Statistics of the International Monetary Fund. We refer the reader to Campbell (2003) for a detailed description of the data. We construct quarterly series of stock market returns, risk-free rates, and per capita consumption growth spanning the early 1970s through the late 1990s for 11 developed countries: Australia, Canada, France, Germany, Italy, Japan, the Netherlands, Sweden, Switzerland, the U.K., and the U.S.. Annual series are available for Sweden (1920-1997), the U.K. (1919-1997) and the U.S. (1891-1997). We also use longer quarterly (1947.2-2019.3) and annual (1891-2018) U.S. data which yield consistent empirical findings.

5.1. Equity Risk Premium

In the finance literature, a large discrepancy between the C-CAPM prediction and the empirical reality is often referred to as a "puzzle". The well-documented equity premium puzzle implies that the observed equity premium can be matched only by assuming a very high coefficient of risk aversion (Mehra and Prescott 1985). Using our international data set, we calibrate the relative risk aversion index for the alternative (STD, FED, SED) representations of the C-CAPM. Our empirical investigation hinges on the computation of sample statistics for return and consumption growth. For each market in our international data set, we present in Table 4 the annualized mean, annualized standard deviation, skewness, kurtosis, and first-order autocorrelation for the net return on the stock index (R_e), the net return on the risk-free asset (R_t^f), and the per capita consumption growth (g).[17]

The third column in Table 5 shows the equity risk premium proxied by the annualized average excess return ($E(eR_e)$). The fourth column gives the covariance ($\text{cov}\,(g, R_e)$) between return and consumption growth. The fifth column presents the integrated SED between return and consumption growth ($\int_{\underline{g}}^{\bar{g}} \text{cov}\,(-(\xi - g)_+, R_e)\, d\xi$), a measure of downside risk quantity discussed in Sections 3 and 4. The sixth column contains the relative risk aversion calibrated values assuming power utility and joint normal distribution for log return and log consumption growth, as per equation (16) in Campbell (2003). The last three columns report the calibrated relative risk aversion ($\gamma^{calibrated}$) for the STD, FED, and SED representations of the C-CAPM in (10-12). For nearly all countries, the computed relative risk aversion values are positive. As argued by Campbell (2003), negative distortions in the calibrated relative risk aversion values for France, Italy, and Switzerland may stem from short-term measurement errors in the corresponding consumption series.

The relative risk aversion values from Campbell's (2003) approach in column 6 of Table 5 are much larger than the ones calibrated from the expectation dependence-based representations. This is expected because the joint normal distribution assumption in Campbell (2003) implies that the pricing equation only depends on the first two moments and comoment. Focusing on columns 7-9 in Table 5, we see a sharp reduction in the calibrated relative risk aversion values when the C-CAPM representation includes the consumption second-degree expectation dependence effect. For quarterly data, the relative risk aversion coefficients implied by the SED representation are roughly 2 to 10 (5 to 95, respectively) times smaller than their FED-implied (STD-implied, respectively) counterparts. For annual data, the reduction ratio for $\gamma^{calibrated}$ ranges from 2 to 6 for the SED representation relative to the FED representation; and 4 to 10 for the SED representation relative to the STD representation. Specifically for quarterly U.S. series, the calibrated risk aversion coefficients from the SED-based formulation of the pricing rule are 19.9 (1947.2–1998.3), 19.3 (1970.1–1998.3), and 16.5 (1947.2–2019.3). The corresponding relative risk aversion values computed from annual data for the U.S. (2.7 for 1891–1997, and 2.8 for 1891–2018) are in line with the numbers often proposed in the asset pricing theory. This also holds for the U.K. (3.6 for 1919–1997) and Sweden (14.6 for 1920–1997).

In a nutshell, the SED representation improves the C-CAPM calibration and delivers reasonable relative risk aversion coefficients. Beyond its theoretical appeal, the concept of expectation dependence can help refine the assessment of risk and bridge the gap between real-world data and the C-CAPM prediction.

5.2. Variance Risk Premium

The variance risk premium (VRP) reflects the premium accrued to bearing the uncertainty surrounding future variance. For a given horizon τ, the VRP is commonly defined as the spread between the realized variance ($\tilde{V}_{t,\tau}$) expected under the risk-neutral probability measure ($E_t^Q(\tilde{V}_{t,\tau}) \equiv V_{t,\tau}^Q$) and its physical counterpart ($E_t^P(\tilde{V}_{t,\tau}) \equiv V_{t,\tau}^P$). The stochastic discount factor maps the physical probability measure (P) to the risk-neutral probability measure (Q), by assigning more weight to "bad" states of the economy characterized by high marginal utility of consumption. Formally,

$$VRP_{t,\tau} = V^Q_{t,\tau} - V^P_{t,\tau}. \tag{18}$$

Bollerslev et al. (2009), and Feunou et al. (2018), among others, build successful models to explain the VRP dynamics. At maturity and under no arbitrage, the spread in (18) equals the terminal payoff to the short leg of a variance swap contract, as discussed in Carr and Wu (2009).[18] The valuation formula of this variance swap contract implies

$$VRP_{t,\tau} = \frac{\mathrm{cov}_t[u'(\tilde{c}_{t+\tau}), \tilde{V}_{t+\tau}]}{E_t u'(\tilde{c}_{t+\tau})}. \tag{19}$$

Given that the risk-neutral expectation of a risky asset return is equal to the risk-free rate ($E^Q_t(\tilde{R}_{t,\tau}) = R^f_t$), the variance risk premium formula in (19) appears as the opposite of the equity risk premium relation in (1), where the return is replaced with the realized variance. Thus, the FED representation of the variance risk premium formula is

$$\frac{VRP_{t,\tau}}{\beta(1+R^f_t)} = -\int_{\underline{g}}^{\bar{g}} \mathrm{FED}(\tilde{V}_{t+\tau}|g_{t+\tau}) F_{\tilde{g}_{t+\tau}}(g_{t+\tau})\,\mathrm{RR}_2(c_t g_{t+\tau})\,\mathrm{MRS}_{c_t g_{t+\tau},c_t}\,\frac{1}{g_{t+\tau}} dg_{t+\tau}, \tag{20}$$

and the SED representation is

$$\begin{aligned} \frac{VRP_{t,\tau}}{\beta(1+R^f_t)} &= -\,\mathrm{cov}_t(\tilde{V}_{t+\tau}, \tilde{g}_{t+\tau})\,\mathrm{RR}_2(c_t\bar{g})\,\mathrm{MRS}_{c_t\bar{g},c_t}\,\frac{1}{\bar{g}} \\ &\quad - \int_{\underline{g}}^{\bar{g}} \mathrm{SED}(\tilde{V}_{t+\tau}|g_{t+\tau})\,\mathrm{RR}_3(c_t g_{t+\tau})\,\mathrm{MRS}_{c_t g_{t+\tau},c_t}\,\frac{1}{g^2_{t+\tau}} dg_{t+\tau}. \end{aligned} \tag{21}$$

From an empirical perspective, constructing VRP series from actual data requires both physical and risk-neutral forecasts of the realized variance. The physical forecast of the realized variance $V_{t,\tau}^{P}$ can be obtained from high frequency stock data as in Andersen et al. (2003), using for instance Corsi's (2009) projections. Following Bakshi and Madan (2000), Bakshi et al. (2003), and Carr et al. (2012), the risk-neutral expectation of the realized variance $V_{t,\tau}^{Q}$ can be extracted from a panel of European options written on the stock.[19] Further details on the construction of these series are presented in Appendix A7.

Because asset price series are monthly, we sample these variables at quarter ends to align them with quarterly consumption series. For the U.S. market, the resulting VRP series spans 1996.3 to 2015.3.[20] To match the most recent quarter in Campbell's (2003) study, we also consider a shorter period ending in 2011.3. Using the alternative representations of the pricing formula (19), (20), and (21) in turn, we calibrate the relative risk aversion coefficient to match the observed VRP in the U.S. market. Table 6 reports summary statistics for the relevant series along with the calibration results. We observe a negative correlation between consumption growth and realized variance, which is in line with the empirical evidence documented by Bollerslev et al. (2009). Looking at the calibrated relative risk aversion, we notice that the SED representation of the consumption-based variance risk pricing formula delivers a realistic value below 10 in the full sample as well as in the shorter sample.

6. Conclusion

We rely on the concept of expectation dependence to derive an alternative representation of the C-CAPM. The proposed alternative formulation of the pricing formula underscores the importance of refining the assessment of the dependence between consumption and asset return. We provide theoretical and empirical arguments to support general measures of risk dependence in the valuation of financial assets. Using both simulated and observed data, we show that accounting for higher degrees of risk dependence and higher orders of risk attitude is key to understanding the variations in asset returns and corresponding premia.

Notes

[1]When $(\tilde{c}_{t+1}, \tilde{R}_{t+1})$ is jointly normally distributed, Stein's (1973) lemma can be applied to compute $\text{cov}_t[u'(\tilde{c}_{t+1}), \tilde{R}_{t+1}] = \text{cov}_t(\tilde{c}_{t+1}, \tilde{R}_{t+1})E_t(u''(\tilde{c}_{t+1})) = \text{cov}_t(\tilde{g}_{t+1}, \tilde{R}_{t+1})E_t(c_t u''(\tilde{c}_{t+1}))$.

[2]Pellerey and Semeraro (2005), and Dachraoui and Dionne (2007) explore portfolio selection problems using quadrant dependence, a concept that is stronger than expectation dependence.

[3]The necessary and sufficient conditions are proven in Appendix A2.

[4]Here, time subscripts are dropped to ease notation without loss of generality.

[5]Theoretically, when the upper bound of the consumption growth is infinite, $u''(c_t\bar{g} \to +\infty)$ in the first term of (4) converges to a finite quantity because u'' is an up-bounded $(u'' < 0)$ and monotone increasing function $(u''' > 0)$.

[6]A relative prudence index $-c_{t+1}u'''(c_{t+1})/u''(c_{t+1})$ is consistent with Kimball's (1990) definition.

[7]Alternatively, one can use higher order generalizations of the Arrow-Pratt absolute risk aversion measure, $-u^{(k)}/u^{(k-1)}$.

[8]Alternative specifications of the marginal distributions of financial series include Harvey and Siddique's (1999) non-central Student's t.

[9]For negatively dependent covariates, $\mathbf{C}(q, 1-q)$ and $\mathbf{C}(1-q, q)$ are used in the definition of quantile dependence.

[10]The Joe-Clayton copula is labelled as the family BB7 in Joe (1997).

[11] The upper (τ_U) and lower (τ_D) tail dependence measures are identical for the copula of an elliptically symmetric distribution. For instance, $\tau_U = \tau_D > 0$ for a Student's t copula. By contrast, $\tau_U = \tau_D = 0$ for a Gaussian copula, implying that the variables are independent in the extreme tails of the joint distribution

(Embrechts, McNeil, and Straumann 2002).

[12] Using exceedance correlation measures, Longin and Solnik 2001, Ang and Chen 2002 document that equity returns are more dependent in bear markets than in bull markets.

[13]Following Campbell (2003), we use a numerical calibration to compute the implied relative risk aversion values, although a generalized method of moments estimation could also be used as in Martin (2013). The computation of integrals are presented at the end of Appendix A5.

[14]As expected, the stronger the correlation ρ, the more accurate the calibrated relative risk aversion value.

[15]The analytical expressions for the skewness and kurtosis of the Hansen's (1994) skew t distribution are given in Appendix A6.

[16]http://scholar.harvard.edu/campbell/data.

[17]The average net returns in Table 4 are slightly higher than the average log returns in Campbell's (2003) study due to Jensen's inequality.

[18]The terminal profit and loss from a long variance swap contract that is held to maturity is given by $V^P_{t,\tau} - V^Q_{t,\tau}$.

[19] We apply the same filters as in Chang et al. (2013). To get a dense set of observations, option prices are mapped into Black and Scholes' (1973) implied volatilities that then are interpolated.

[20]We use OptionMetrics S&P 500 index option data from September 03, 1996 to August 31, 2015 to construct the variance risk premium series.

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Table 1 Descriptive Statistics and skew t marginal distribution parameter estimates (U.S. 1947.2-2019.3)

	Descriptive Statistics					Skew t Parameter Estimates			
	Mean	Std. Dev.	Skewness	Kurtosis	Autocorr.	ν	$SE(\nu)$	λ	$SE(\lambda)$
R	0.021	0.077	-0.584	4.030	0.072	6.824	2.363	-0.179	0.085
g	1.005	0.005	-0.466	4.385	0.314	4.715	0.956	-0.110	0.074

Notes: This table presents the nonannualized quarterly descriptive statistics for the return (R) and consumption growth series (g). The last columns present parameter estimates and standard errors (SE) from skew t models for marginal distributions of standardized innovations in return ($R^* = (R - E(R))/\sigma(R)$) and consumption growth ($g^* = (g - E(g))/\sigma(g)$).

Table 2 Simulations with Different Dependence Structures

	Dependence structure for simulation			$\gamma^{calibrated}$						Reduction			
	Tail		Corr	STD		FED		SED		1-FED/STD		1-SED/STD	
	τ_U	τ_D	ρ	*bias*	*rmse*	*bias*	*rmse*	*bias*	*rmse*	*bias*	*rmse*	*bias*	*rmse*
Relative risk aversion $\gamma_{sim} = 5$													
Panel A: High correlation													
simH1	0.20	0.75	0.75	0.67	1.11	0.30	0.35	0.16	0.19	0.56	0.69	0.76	0.83
simH2	0.75	0.20	0.75	0.66	1.06	0.46	0.54	0.25	0.29	0.31	0.49	0.63	0.72
simH3	0.55	0.55	0.75	0.64	1.07	0.37	0.43	0.22	0.27	0.43	0.60	0.65	0.75
Panel B: Medium correlation													
simM1	0.20	0.45	0.50	0.75	1.38	0.33	0.39	0.16	0.21	0.57	0.72	0.78	0.85
simM2	0.45	0.20	0.50	0.92	1.67	0.61	0.74	0.32	0.41	0.34	0.56	0.65	0.75
simM3	0.30	0.30	0.50	0.74	1.35	0.40	0.49	0.23	0.31	0.45	0.64	0.68	0.77
Panel C: Low correlation													
simL1	5.15×10^{-5}	0.20	0.25	1.00	2.34	0.54	0.79	0.24	0.35	0.46	0.66	0.76	0.85
simL2	0.20	5.15×10^{-5}	0.25	1.02	2.89	0.92	1.66	0.29	0.44	0.10	0.43	0.71	0.85
simL3	0.10	0.10	0.25	0.79	2.12	0.61	1.24	0.26	0.38	0.22	0.41	0.68	0.82
Relative risk aversion $\gamma_{sim} = 10$													
Panel D: High correlation													
simH1	0.20	0.75	0.75	1.39	2.34	0.59	0.70	0.34	0.37	0.58	0.70	0.76	0.84
simH2	0.75	0.20	0.75	1.96	3.12	0.89	1.07	1.45	1.54	0.54	0.66	0.26	0.51
simH3	0.55	0.55	0.75	1.29	2.21	0.73	0.88	0.71	0.77	0.43	0.60	0.45	0.65
Panel E: Medium correlation													
simM1	0.20	0.45	0.50	1.61	2.92	0.65	0.79	0.40	0.46	0.60	0.73	0.75	0.84
simM2	0.45	0.20	0.50	1.85	3.38	1.24	1.48	1.53	1.70	0.33	0.56	0.17	0.50
simM3	0.30	0.30	0.50	1.51	2.79	0.82	1.00	0.83	0.93	0.45	0.64	0.45	0.67
Panel F: Low correlation													
simL1	5.15×10^{-5}	0.20	0.25	1.92	4.07	0.99	1.37	0.53	0.68	0.48	0.66	0.72	0.83
simL2	0.20	5.15×10^{-5}	0.25	2.09	5.02	1.77	2.66	0.77	1.11	0.15	0.47	0.63	0.78
simL3	0.10	0.10	0.25	1.55	3.37	1.16	1.59	0.87	1.06	0.25	0.53	0.44	0.68

Notes: This table presents the results for simulated return and consumption growth series. We conduct two sets of simulations assuming relative risk aversion values of $\gamma_{sim} = 5$ (Panels A, B, C) and $\gamma_{sim} = 10$ (Panels D, E, F). The upper and lower tail dependence measures are τ_U and τ_D, and ρ is the corresponding coefficient of linear correlation (Corr) as in Patton (2013). Larger values of τ_U (resp. τ_D) induce a stronger upper (resp. lower) tail dependence. We consider various dependence structures for the simulations. Specifically, simulations are performed assuming (1) a stronger left tail dependence ($\tau_U < \tau_D$), (2) a stronger right tail dependence ($\tau_U > \tau_D$), and (3) a symmetric tail dependence ($\tau_U = \tau_D$). For comparability, tail dependence parameters in each panel are chosen to imply the same level of linear correlation. Tail dependence parameter values in Panels A and D (resp. B and E, and C and F) imply a high (resp. medium, and low) linear correlation between the series. Note that tail dependence parameters for the simulation simL1 in Panels C and F are set close to the estimated values in the data. We use skew t distributions estimated in Table 1 for the marginals. Equity risk premium series are generated from joint asset return and consumption growth simulated series using a fixed value of relative risk aversion γ_{sim}. For each simulated series of $T = 250$ observations, we perform $N_{sim} = 1000$ replications. Simulated series (equity risk premium, asset return, and consumption growth) are then used to calibrate the relative risk aversion level using the standard (STD), the first-order expectation dependence (FED), and the second-order expectation dependence (SED) representations in (10), (11), and (12). We report the $bias = N_{sim}^{-1} \sum_{i=1}^{N_{sim}} (\gamma_i^{calibrated} - \gamma_{sim})$ and the root-mean-squared error denoted by $rmse = \sqrt{N_{sim}^{-1} \sum_{i=1}^{N_{sim}} (\gamma_i^{calibrated} - \gamma_{sim})^2}$ as performance metrics for the different representations. The last columns show the reduction in *bias* and *rmse* of FED and SED representations relative to the standard representation.

Table 3 Simulations with Different Autocorrelation, Skewness, and Kurtosis Parameters

	Distribution parameters			$\gamma^{calibrated}$						Reduction			
	Autocorr	Skewness	Kurtosis	STD		FED		SED		1-FED/STD		1-SED/STD	
	(ϕ_R, ϕ_g)	(λ_R, λ_g)	(ν_R, ν_g)	*bias*	*rmse*	*bias*	*rmse*	*bias*	*rmse*	*bias*	*rmse*	*bias*	*rmse*
Relative risk aversion $\gamma_{sim} = 5$													
simP1	$(0.2, 0.6)$	$(-0.2, -0.1)$	$(6.8, 4.7)$	1.79	4.60	0.89	1.50	0.51	0.83	0.50	0.67	0.72	0.82
simP2	$(0.0, 0.0)$	$(-0.2, -0.1)$	$(6.8, 4.7)$	1.79	3.89	0.74	1.01	0.33	0.50	0.59	0.74	0.81	0.87
simP3	$(0.2, 0.6)$	$(0.0, 0.0)$	$(6.8, 4.7)$	1.91	8.92	1.83	8.08	1.72	7.61	0.04	0.09	0.10	0.15
simP4	$(0.2, 0.6)$	$(-0.2, -0.1)$	$(150, 100)$	1.10	4.20	0.98	3.52	0.90	3.12	0.11	0.16	0.18	0.26
simP5	$(0.0, 0.0)$	$(0.0, 0.0)$	$(150, 100)$	1.20	2.97	1.25	3.05	1.24	2.92	-0.05	-0.03	-0.03	0.02

Notes: We use skew t parameters estimated in Table 1 to simulate the distributions of return and consumption growth, as in (13). The joint distribution dependence is built using SJC copula parameters set to their empirical estimates $\tau_U = 5.15 \times 10^{-5}, \tau_D = 0.20$, implying $\rho = 0.25$. Equity risk premium series are generated from joint asset return and consumption growth simulated series using a fixed value of relative risk aversion $\gamma_{sim} = 5$. For each simulated series of $T = 250$ observations, we perform $N_{sim} = 1000$ replications. In simP1, the skewness and kurtosis parameters are set to match their empirical estimates. The chosen parameter values $(\phi_R, \phi_g) = (0.2, 0.6)$, $(\lambda_R, \lambda_g) = (-0.2, -0.1)$ and $(\nu_R, \nu_g) = (6.8, 4.7)$ imply the following skewness $(s_R, s_g) = (-0.6, -0.3)$ and kurtosis $(k_R, k_g) = (5.4, 7.3)$. See Appendix A6 for the skewness and kurtosis formulas of the Hansen's (1994) skew t distribution. simP2 reports the simulations when there is no autocorrelation, $(\phi_R, \phi_g) = (0, 0)$. The implied skewness $(s_R, s_g) = (-0.6, -0.5)$ and kurtosis $(k_R, k_g) = (5.6, 12.1)$ in simP2 are slightly more pronounced than in simP1 because the positive empirical autocorrelations are weak. simP2 presents the simulations assuming zero skewness. In simP3, the implied skewness and kurtosis are $(s_R, s_g) = (0, 0)$ and $(k_R, k_g) = (5.0, 7.0)$. simP4 investigates the case with almost no excess kurtosis, that is when we set high values for the parameters $(\nu_R, \nu_g) = (150, 100)$, implying $(s_R, s_g) = (-0.3, -0.1)$ and $(k_R, k_g) = (3.1, 3.0)$. Note that the kurtosis gets closer to 3, as $\nu \to \infty$. simP5 assumes no autocorrelation $(\phi_R, \phi_g) = (0, 0)$, no skewness $(s_R, s_g) = (0, 0)$, and almost no excess kurtosis $(k_R, k_g) = (3.0, 3.1)$. The simulated series (equity risk premium, asset return, and consumption growth) are used to calibrate the relative risk aversion level using the standard (STD), the first-order expectation dependence (FED), and the second-order expectation dependence (SED) representations in (10), (11), and (12). We report the $bias = N_{sim}^{-1} \sum_{i=1}^{N_{sim}} (\gamma_i^{calibrated} - \gamma_{sim})$ and the root-mean-squared error denoted by $rmse = \sqrt{N_{sim}^{-1} \sum_{i=1}^{N_{sim}} (\gamma_i^{calibrated} - \gamma_{sim})^2}$ as performance metrics for the different representations. The last columns show the reduction in *bias* and *rmse* of FED and SED representations relative to the standard representation.

Table 4 International Stock, T-Bill Returns, and Consumption Growth

Country	Sample period	$E(R_e)$	$\sigma(R_e)$	$s(R_e)$	$k(R_e)$	$\phi(R_e)$	$E(R^f)$	$\sigma(R^f)$	$s(R^f)$	$k(R^f)$	$\phi(R^f)$	$E(g-1)$	$\sigma(g)$	$s(g)$	$k(g)$	$\phi(g)$
USA	1947.2-1998.4	9.381	15.476	-0.606	4.324	0.092	0.907	1.744	-0.711	7.090	0.508	1.974	1.076	-0.459	3.989	0.216
AUL	1970.1-1999.1	6.005	21.401	-0.875	5.107	0.004	2.091	2.528	-0.806	3.974	0.647	2.125	2.060	-0.648	3.427	-0.325
CAN	1970.1-1999.2	6.931	16.940	-0.587	4.198	0.070	2.739	1.863	-0.559	3.247	0.667	2.107	1.980	-0.082	3.182	0.106
FR	1973.2-1998.4	11.836	23.263	-0.213	4.180	0.054	2.741	1.847	-0.291	2.794	0.710	1.277	2.830	-4.143	27.777	0.029
GER	1978.4-1997.4	11.899	19.195	-1.058	6.348	0.124	3.238	1.160	-0.146	3.123	0.347	1.714	2.450	0.613	5.765	-0.326
ITA	1971.2-1998.2	6.893	28.210	0.839	5.045	0.071	2.418	2.843	-1.026	3.767	0.693	2.220	1.709	-0.052	3.362	0.284
JAP	1970.2-1999.1	7.105	21.668	-0.204	3.463	0.031	1.417	2.287	-1.468	7.464	0.478	3.251	2.569	-0.263	11.112	-0.276
NTH	1977.4-1998.4	16.163	17.183	-0.808	5.623	-0.024	3.551	1.547	-0.235	2.780	-0.173	1.798	2.506	0.453	5.963	-0.216
SWD	1970.1-1999.3	13.642	24.150	0.086	4.129	0.060	2.040	2.841	-0.367	3.470	0.260	0.980	1.855	-0.646	5.933	-0.266
SWT	1982.2-1999.1	16.293	21.135	-0.845	5.992	-0.133	1.406	1.499	-0.776	4.448	0.243	0.547	2.121	0.556	3.606	-0.398
UK	1970.1-1999.2	10.534	22.259	1.267	12.156	0.036	1.346	2.932	-1.578	8.396	0.481	2.240	2.514	-0.364	7.496	-0.005
USA	1970.1-1998.4	8.506	17.278	-0.572	4.0304	0.061	1.502	1.694	0.420	3.653	0.570	1.821	0.910	-0.482	3.992	0.374
USA	1947.2-2019.4	8.282	15.448	-0.589	4.041	0.072	0.544	1.780	0.003	7.047	0.374	1.880	0.996	-0.468	4.366	0.314
SWD	1920-1998	8.519	19.399	0.206	2.962	0.057	2.391	6.665	2.022	11.451	0.667	1.782	2.863	-0.340	4.907	0.173
UK	1919-1998	10.196	23.620	0.741	5.925	-0.074	1.405	6.011	1.08	8.094	0.585	1.517	2.869	0.086	4.751	0.284
USA	1891-1998	8.622	18.843	-0.210	2.816	0.003	2.382	9.110	0.857	8.169	0.400	1.837	3.244	-0.367	4.540	-0.129
USA	1891-2019	8.164	18.430	-0.264	2.950	-0.015	1.947	8.423	1.051	9.538	0.407	1.811	3.039	-0.408	5.043	-0.119

Notes: This table displays the annualized mean (E in percent), annualized standard deviation (σ in percent), skewness (s), kurtosis (k), and first-order autocorrelation (ϕ) for the net return on the stock market index (R_e), the net return on the risk-free asset (R^f), and the per capita consumption growth (g). Time subscripts are dropped to ease notation.

Table 5 Equity Risk Premium

Country	Sample period	$E(eR_e)$	$\text{cov}(g, R_e)$	$\int_{\underline{g}}^{\bar{g}} \text{cov}(-(\xi - g)_+, R_e)\, d\xi$	$\gamma^{calibrated}$ Campbell (2003)	STD	FED	SED
USA	1947.2-1998.3	8.474	3.863	0.058	211.055	142.149	131.048	19.893
AUL	1970.1-1998.4	3.913	7.352	0.255	53.434	42.463	33.231	7.640
CAN	1970.1-1999.1	4.192	6.146	0.087	116.196	82.469	73.663	23.560
FR	1973.2-1998.3	9.095	-6.308	-0.613	< 0	< 0	< 0	< 0
GER	1978.4-1997.3	8.661	0.728	0.108	639.433	115.203	67.040	6.815
ITA	1971.2-1998.1	4.475	-1.602	-0.055	< 0	< 0	< 0	< 0
JAP	1970.2-1998.4	5.688	6.167	0.581	82.866	43.931	33.925	4.791
NTH	1977.4-1998.3	12.613	0.565	0.005	900.824	98.072	84.161	31.621
SWD	1970.1-1999.2	11.602	0.447	0.071	1670.266	217.570	119.041	85.685
SWT	1982.2-1998.4	14.887	-5.646	-0.298	< 0	< 0	< 0	< 0
UK	1970.1-1999.1	9.188	6.464	0.189	139.240	62.796	60.918	12.408
USA	1970.1-1998.3	7.004	4.582	0.063	140.920	115.978	97.116	19.348
USA	1947.2-2019.3	7.737	3.901	0.068	192.768	134.160	117.783	16.529
SWD	1920-1997	6.129	10.587	0.513	53.477	45.505	31.782	14.574
UK	1919-1997	8.791	24.638	2.476	34.216	26.496	20.813	3.566
USA	1891-1997	6.241	26.840	2.101	24.126	19.051	15.573	2.728
USA	1891-2018	6.218	24.864	2.054	24.872	19.904	16.387	2.818

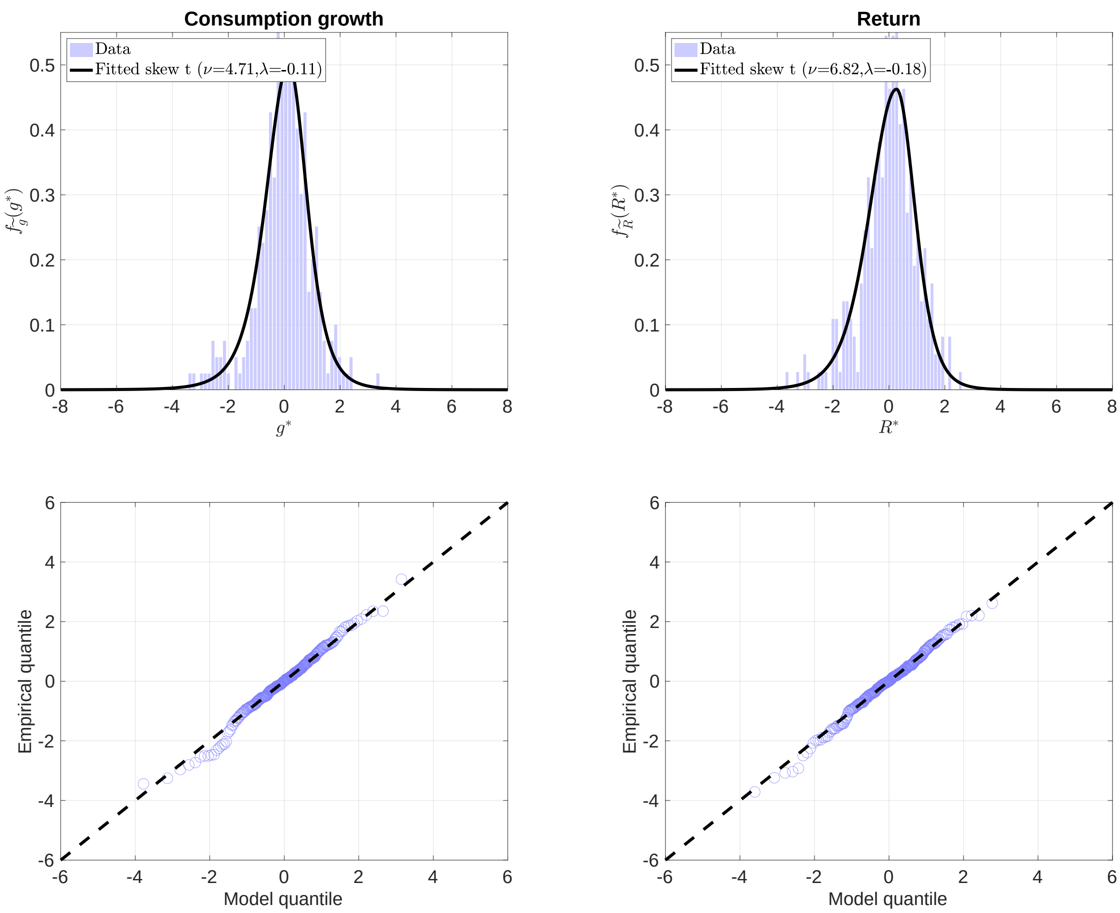
Notes: The third column in this table displays the annualized average excess net return ($E(eR_e) = E(R_e - R^f)$ in percent). The fourth column presents the annualized covariance between return and consumption growth ($\text{cov}(g, R_e)$ in basis point). The fifth column gives the annualized integrated SED between return and consumption growth ($\int_{\underline{g}}^{\bar{g}} \text{cov}(-(\xi - g)_+, R_e)\, d\xi$ in basis point). The sixth column presents the relative risk aversion calibrated values assuming log return and log consumption growth are jointly normally distributed, as per equation (16) in Campbell (2003). The last three columns report the calibrated relative risk aversion ($\gamma^{calibrated}$) for the standard (STD), the first-order expectation dependence (FED), and the second-order expectation dependence (SED) representations of the consumption-based equity risk pricing formula. Time subscripts are dropped to ease notation.

Table 6 Variance Risk Premium for the U.S.

								$\gamma^{calibrated}$		
Sample period	$E(VRP)$	$\sigma(VRP)$	$E(V)$	$\sigma(V)$	$E(g-1)$	$\sigma(g)$	corr (g, V)	STD	FED	SED
1996.3-2011.3	2.382	1.492	3.755	4.645	1.791	0.882	-0.349	173.696	150.726	9.237
1996.3-2015.3	2.155	1.922	3.518	4.375	2.012	0.883	-0.330	165.691	148.697	8.731

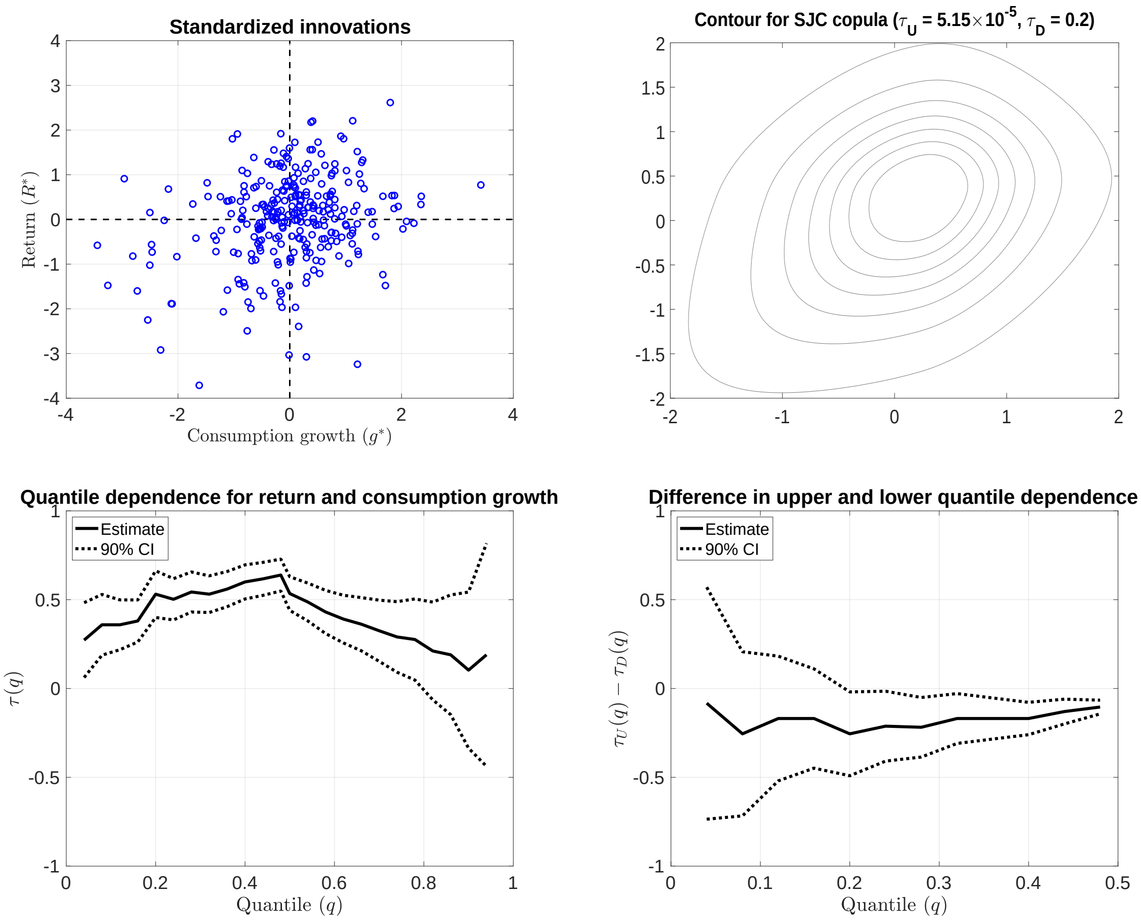
Notes: The annualized mean (E) and standard deviation (σ) of the variance risk premium (VRP), the realized variance (V), and the consumption growth (g) are expressed in percentages (%). Price-related series (VRP, V) are monthly. Thus, we sample these variables at quarter ends to align them with quarterly consumption series (g). The last three columns report the calibrated relative risk aversion ($\gamma^{calibrated}$) for the standard (STD), the first-order expectation dependence (FED), and the second-order expectation dependence (SED) representations of the consumption-based variance risk pricing formula.

Figure 1 Marginal distributions (U.S. 1947.2-2019.3)



The top panel shows the histograms of the standardized innovations in return and consumption growth along with the corresponding fitted skew t densities. The bottom panel presents the QQ plots.

Figure 2 Joint distribution (U.S. 1947.2-2019.3)



The top left panel shows the scatter plot of the standardized innovations in return and consumption growth. The top right panel exhibits the fitted symmetrized Joe-Clayton copula with skew t marginal distributions estimated in Table 1. The bottom panel presents the tail dependence estimates along with the difference between upper and lower tail dependence estimates. The dotted lines represents the 90% confidence intervals.

A. Appendix

A1. Derivation of Equations (2) and (4)

We start with the proof of (2). From Theorem 1 in Cuadras (2002), we have

$$\begin{aligned} & cov_t[u'(\tilde{c}_{t+1}), \tilde{R}_{t+1}] \qquad (22) \\ = & \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} [F(c_{t+1}, R_{t+1}) - F_{\tilde{c}_{t+1}}(c_{t+1}) F_{\tilde{R}_{t+1}}(R_{t+1})] u''(c_{t+1}) dR_{t+1} dc_{t+1} \\ = & \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} [F_{\tilde{R}_{t+1}}(R_{t+1}|\tilde{c}_{t+1} \leq c_{t+1}) - F_{\tilde{R}_{t+1}}(R_{t+1})] dR_{t+1} \right] F_{\tilde{c}_{t+1}}(c_{t+1}) u''(c_{t+1}) dc_{t+1} \end{aligned}$$

From Lemma 1 in Tesfatsion (1976), we can write

$$\int_{-\infty}^{+\infty} [F_{\tilde{R}_{t+1}}(R_{t+1}|\tilde{c}_{t+1} \leq c_{t+1}) - F_{\tilde{R}_{t+1}}(R_{t+1})] dR_{t+1} = E\tilde{R}_{t+1} - E(\tilde{R}_{t+1}|\tilde{c}_{t+1} \leq c_{t+1}). \quad (23)$$

By definition, the random future consumption $\tilde{c}_{t+1}$ equals $c_t \tilde{g}_{t+1}$, where $\tilde{g}_{t+1}$ denotes the consumption growth and $c_t > 0$ is the current consumption. Similarly, for a given realization of future consumption, we have $c_{t+1} = c_t g_{t+1}$. Because $c_t > 0$, $\tilde{c}_{t+1} \leq c_{t+1}$ is equivalent to $\tilde{g}_{t+1} \leq g_{t+1}$. Thus, the marginal distribution of consumption $F_{\tilde{c}_{t+1}}(c_{t+1}) = Prob(\tilde{c}_{t+1} \leq c_{t+1}) = Prob(\tilde{g}_{t+1} \leq g_{t+1}) = F_{\tilde{g}_{t+1}}(g_{t+1})$. Substituting (23) in (22), we get (2) from Theorem 3.1 in Wright (1987):

$$\begin{aligned} cov_t[u'(c_t \tilde{g}_{t+1}), \tilde{R}_{t+1}] & \equiv cov_t[u'(\tilde{c}_{t+1}), \tilde{R}_{t+1}] \qquad (24) \\ & = \int_{-\infty}^{+\infty} [E\tilde{R}_{t+1} - E(\tilde{R}_{t+1}|\tilde{c}_{t+1} \leq c_{t+1})] F_{\tilde{c}_{t+1}}(c_{t+1}) u''(c_{t+1}) dc_{t+1} \\ & = \int_{-\infty}^{+\infty} [E\tilde{R}_{t+1} - E(\tilde{R}_{t+1}|\tilde{g}_{t+1} \leq g_{t+1})] F_{\tilde{g}_{t+1}}(g_{t+1}) u''(c_t g_{t+1})(c_t dg_{t+1}) \\ & = \int_{\underline{g}}^{\bar{g}} \text{FED}(\tilde{R}_{t+1}|g_{t+1}) c_t u''(c_t g_{t+1}) F_{\tilde{g}_{t+1}}(g_{t+1}) dg_{t+1}. \end{aligned}$$

Next, we derive (4). From (24) have the following equivalence:

$$\begin{aligned}
\text{cov}_t[u'(c_t\tilde{g}_{t+1}),\tilde{R}_{t+1}] &= \int_{\underline{g}}^{\bar{g}} \text{FED}(\tilde{R}_{t+1}|g_{t+1})c_t u''(c_t g_{t+1})F_{\tilde{g}_{t+1}}(g_{t+1})dg_{t+1}, \qquad (25)\\
&= \int_{-\infty}^{+\infty} c_t u''(c_t g_{t+1})[E\tilde{R}_{t+1} - E(\tilde{R}_{t+1}|\tilde{g}_{t+1} \leq g_{t+1})]F_{\tilde{g}_{t+1}}(g_{t+1})dg_{t+1}\\
&\equiv \int_{-\infty}^{+\infty} c_t u''(c_t g_{t+1})d\left(\int_{-\infty}^{g_{t+1}} [E\tilde{R}_{t+1} - E(\tilde{R}_{t+1}|\tilde{g}_{t+1} \leq s)]F_{\tilde{g}_{t+1}}(s)ds\right),\\
&= c_t u''(c_t g_{t+1})\int_{\underline{g}}^{g_{t+1}} [E\tilde{R}_{t+1} - E(\tilde{R}_{t+1}|\tilde{g}_{t+1} \leq s)]F_{\tilde{g}_{t+1}}(s)ds|_{\underline{g}}^{\bar{g}}
\end{aligned}$$

Following Li (2011), we apply integration by parts to the right-hand side term in (25). We obtain

$$\begin{aligned}
\text{cov}_t[u'(c_t\tilde{g}_{t+1}),\tilde{R}_{t+1}] &= c_t u''(c_t\bar{g})\int_{\underline{g}}^{\bar{g}} [E\tilde{R}_{t+1} - E(\tilde{R}_{t+1}|\tilde{g}_{t+1} \leq s)]F_{\tilde{g}_{t+1}}(s)ds\\
&\quad - \int_{\underline{g}}^{\bar{g}}\int_{\underline{g}}^{g_{t+1}} [E\tilde{R}_{t+1} - E(\tilde{R}_{t+1}|\tilde{g}_{t+1} \leq s)]F_{\tilde{g}_{t+1}}(s)ds c_t^2 u'''(c_t g_{t+1})dg_{t+1},\\
&= c_t u''(c_t\bar{g})\,\text{cov}_t(\tilde{R}_{t+1},\tilde{g}_{t+1}) - \int_{\underline{g}}^{\bar{g}} \text{SED}(\tilde{R}_{t+1}|g_{t+1})c_t^2 u'''(c_t g_{t+1})dg_{t+1}.
\end{aligned}$$

A2. Proof of $E_t\tilde{R}_{t+1} - R_t^f > 0 \iff \text{FED}(\tilde{R}_{t+1}|g_{t+1}) > 0$

Note that the risk premium $E_t\tilde{R}_{t+1} - R_t^f$ and $(E_t\tilde{R}_{t+1} - R_t^f)/\beta(1 + R_t^f)$ have the same sign because $\beta(1+R_t^f) > 0$. Thus, the sufficient condition $E_t\tilde{R}_{t+1} - R_t^f > 0 \Leftarrow \text{FED}(\tilde{R}_{t+1}|g_{t+1}) > 0$ is directly obtained from (3).

We prove the necessary condition $E_t\tilde{R}_{t+1} - R_t^f > 0 \Rightarrow \text{FED}(\tilde{R}_{t+1}|g_{t+1}) > 0$ by contradiction. Assume, for sake of contradiction, that $\text{FED}(\tilde{R}_{t+1}|g_{t+1}) < 0$ for g_{t+1}^0. Given that $c_t > 0$, this assumption is equivalent to $\text{FED}(\tilde{R}_{t+1}|c_{t+1}) < 0$ for c_{t+1}^0, where $c_{t+1} = c_t g_{t+1}$ and $c_{t+1}^0 = c_t g_{t+1}^0$. Because of the continuity of the first-order expectation dependence, we have

$\text{FED}(\tilde{R}_{t+1}|c^0_{t+1}) < 0$ in an interval [a,b]. We choose the utility function

$$\bar{u}(c) = \begin{cases} \alpha c - e^{-a} & c < a \\ \alpha c - e^{-c} & a \le c \le b \\ \alpha c - e^{-b} & c > b, \end{cases} \tag{26}$$

where $\alpha > 0$. Then

$$\bar{u}'(c) = \begin{cases} \alpha & c < a \\ \alpha + e^{-c} & a \le c \le b \\ \alpha & c > b \end{cases} \tag{27}$$

and

$$\bar{u}''(c) = \begin{cases} 0 & c < a \\ -e^{-c} & a \le c \le b \\ 0 & c > b. \end{cases} \tag{28}$$

Therefore,

$$\begin{aligned} \frac{E_t\tilde{R}_{t+1} - R^f_t}{\beta(1+R^f_t)} &= \int_a^b \text{FED}(\tilde{R}_{t+1}|g_{t+1})F_{\tilde{g}_{t+1}}(g_{t+1})\frac{e^{-c_t g_{t+1}}}{\bar{u}'(c_t)}\,(c_t dg_{t+1}) \\ &= \int_a^b \text{FED}(\tilde{R}_{t+1}|g_{t+1})F_{\tilde{g}_{t+1}}(g_{t+1})\frac{e^{-c_{t+1}}}{\bar{u}'(c_t)}dc_{t+1} \\ &< 0. \end{aligned} \tag{29}$$

This is a contradiction.

A3. C-CAPM in Price Form

Alternatively, the C-CAPM pricing rule in (1) can be cast in price form

$$p_t = \frac{E_t \tilde{x}_{t+1}}{1 + R_t^f} + \beta \frac{\text{cov}_t[u'(\tilde{c}_{t+1}), \tilde{x}_{t+1}]}{u'(c_t)}, \tag{30}$$

where the asset's price p_t is determined by the dependence between the marginal utility of consumption $u'(\tilde{c}_{t+1})$ and the asset's payoff $\tilde{x}_{t+1}$. By exploiting the formula of first-order expectation dependence (FED) in (2), we can rewrite (1) in price form as

$$\begin{aligned} p_t &= \frac{E_t \tilde{x}_{t+1}}{1 + R_t^f} - \beta \int_{\underline{g}}^{\bar{g}} \text{FED}(\tilde{x}_{t+1} | g_{t+1}) F_{\tilde{g}_{t+1}}(g_{t+1}) \left(-c_t \frac{u''(c_t g_{t+1})}{u'(c_t)} \right) dg_{t+1} \\ &= \underbrace{\frac{E_t \tilde{x}_{t+1}}{1 + R_t^f}}_{\text{risk-free present value effect}} \underbrace{- \beta \int_{\underline{g}}^{\bar{g}} \text{FED}(\tilde{x}_{t+1} | g_{t+1}) F_{\tilde{g}_{t+1}}(g_{t+1}) \, \text{RR}_2(c_t g_{t+1}) \, \text{MRS}_{c_t g_{t+1}, c_t} \frac{1}{g_{t+1}} dg_{t+1}}_{\text{first-degree expectation dependence effect}} \end{aligned} \tag{31}$$

The asset price formula in (31) involves two terms. The first term on the right-hand side measures the risk-free present value effect. This term captures the direct effect of the risk-free present expected payoff, which characterizes the asset's price for a risk-neutral representative agent. In line with (3), the second term on the right-hand side of (31) reflects the first-degree expectation dependence effect or the FED effect. This term involves the time discount factor, the first-degree expectation dependence between the asset's payoff and consumption, the Arrow–Pratt relative risk aversion index, and the intertemporal marginal rate of substitution. The sign of the first-degree expectation dependence indicates whether changes in consumption tend to reinforce (positive FED) or counteract (negative FED) the random fluctuations of the asset's payoff. In this representation, an asset's price is lowered (raised) if and only if its payoff is positively (negatively) first-degree expectation dependent with consumption.

Using integration by parts and second-order expectation dependence (SED) as in Li

(2011), (31) can be restated as

$$
\begin{aligned}
p_t &= \frac{E_t\tilde{x}_{t+1}}{1+R_t^f} - \beta\,\mathrm{cov}_t(\tilde{x}_{t+1},\tilde{g}_{t+1})\left(-c_t\frac{u''(c_t\bar{g})}{u'(c_t)}\right) - \beta\int_{\underline{g}}^{\bar{g}}\mathrm{SED}(\tilde{x}_{t+1}|g_{t+1})\left(c_t^2\frac{u'''(c_tg_{t+1})}{u'(c_t)}\right)dg_{t+1} \\
&= \frac{E_t\tilde{x}_{t+1}}{1+R_t^f} - \underbrace{\beta\,\mathrm{cov}_t(\tilde{x}_{t+1},\tilde{g}_{t+1})\,\mathrm{RR}_2(c_t\bar{g})\,\mathrm{MRS}_{c_t\bar{g},c_t}\,\frac{1}{\bar{g}}}_{\text{covariance effect}} \\
&\quad \underbrace{-\,\beta\int_{\underline{g}}^{\bar{g}}\mathrm{SED}(\tilde{x}_{t+1}|g_{t+1})\,\mathrm{RR}_3(c_tg_{t+1})\,\mathrm{MRS}_{c_tg_{t+1},c_t}\,\frac{1}{g_{t+1}^2}dg_{t+1},}_{\text{second-degree expectation dependence effect}} \qquad (32)
\end{aligned}
$$

where $\mathrm{RR}_3(c_{t+1}) = c_{t+1}^2u'''(c_{t+1})/u'(c_{t+1})$ is an index of relative downside risk aversion (Modica and Scarsini, 2005). The last term on the right-hand side of (32) shows the SED effect, which involves the SED risk, the intensity of downside risk aversion, and the marginal rate of intertemporal substitution, integrated over the support of the consumption distribution.

A4. Expectation Dependence-based Pricing with Absolute Risk Indexes

We rewrite the expectation dependence-based pricing formulas in terms of absolute risk indexes assuming that the representative investor is equipped with a constant absolute risk aversion (CARA) utility function. The CARA utility assumption may be theoretically useful to make some qualitative statements. It is straightforward to restate (31) and (3) as

$$
\begin{aligned}
p_t &= \frac{E_t\tilde{x}_{t+1}}{1+R_t^f} - \beta\int_{\underline{c}}^{\bar{c}}\mathrm{FED}(\tilde{x}_{t+1}|c_{t+1})F_{\tilde{c}_{t+1}}(c_{t+1})[-\frac{u''(c_{t+1})}{u'(c_t)}]dc_{t+1}, \\
&= \underbrace{\frac{E_t\tilde{x}_{t+1}}{1+R_t^f}}_{\text{risk-free present value effect}} \underbrace{-\,\beta\int_{\underline{c}}^{\bar{c}}\mathrm{FED}(\tilde{x}_{t+1}|c_{t+1})F_{\tilde{c}_{t+1}}(c_{t+1})\,\mathrm{AR}_2(c_{t+1})\,\mathrm{MRS}_{c_{t+1},c_t}\,dc_{t+1},}_{\text{first-degree expectation dependence effect}} \qquad (33)
\end{aligned}
$$

and

$$
\frac{E_t\tilde{R}_{t+1}-R_t^f}{\beta(1+R_t^f)} = \int_{\underline{c}}^{\bar{c}}\underbrace{\mathrm{FED}(\tilde{R}_{t+1}|c_{t+1})F_{\tilde{c}_{t+1}}(c_{t+1})}_{\text{consumption risk effect}}\,\mathrm{AR}_2(c_{t+1})\,\mathrm{MRS}_{c_{t+1},c_t}\,dc_{t+1}, \qquad (34)
$$

where $\mathrm{AR}_2(c_{t+1}) = -u''(c_{t+1})/u'(c_{t+1})$ refers to the Arrow–Pratt measure of absolute risk aversion. Similarly, we can rewrite (32) and (5) as

$$\begin{aligned} p_t &= \frac{E_t\tilde{x}_{t+1}}{1+R_t^f} - \beta\,\mathrm{cov}_t(\tilde{x}_{t+1},\tilde{c}_{t+1})\left(-\frac{u''(\bar{c})}{u'(c_t)}\right) - \beta\int_{\underline{c}}^{\bar{c}} \mathrm{SED}(\tilde{x}_{t+1}|c_{t+1})\left(\frac{u'''(c_{t+1})}{u'(c_t)}\right) dc_{t+1}, \\ &= \frac{E_t\tilde{x}_{t+1}}{1+R_t^f} - \underbrace{\beta\,\mathrm{cov}_t(\tilde{x}_{t+1},\tilde{c}_{t+1})\,\mathrm{AR}_2(\bar{c})\,\mathrm{MRS}_{\bar{c},c_t}}_{\text{covariance effect}} \\ &\quad \underbrace{-\beta\int_{\underline{c}}^{\bar{c}} \mathrm{SED}(\tilde{x}_{t+1}|c_{t+1})\,\mathrm{AR}_3(c_{t+1})\,\mathrm{MRS}_{c_{t+1},c_t}\,dc_{t+1}}_{\text{second-degree expectation dependence effect}}, \end{aligned} \tag{35}$$

and

$$\begin{aligned} \frac{E_t\tilde{R}_{t+1}-R_t^f}{\beta(1+R_t^f)} &= \underbrace{\mathrm{cov}_t(\tilde{R}_{t+1},\tilde{c}_{t+1})}_{\text{covariance risk}}\mathrm{AR}_2(\bar{c})\,\mathrm{MRS}_{\bar{c},c_t} \\ &\quad + \int_{\underline{c}}^{\bar{c}} \underbrace{\mathrm{SED}(\tilde{R}_{t+1}|c_{t+1})}_{\text{SED risk}}\mathrm{AR}_3(c_{t+1})\,\mathrm{MRS}_{c_{t+1},c_t}\,dc_{t+1}, \end{aligned} \tag{36}$$

where $\mathrm{AR}_3(c_{t+1}) = u'''(c_{t+1})/u'(c_{t+1})$ denotes an absolute index of prudence as in Modica and Scarsini (2005), Crainich and Eeckhoudt (2008), and Denuit and Eeckhoudt (2010). Kimball (1990) uses $-u'''(c_{t+1})/u''(c_{t+1})$ as an alternative definition of absolute prudence.

A5. Details on the Comparison of Alternative Representations

A5.1. Marginal Rate of Substitution

The reciprocal of the conditional expectation of the intertemporal marginal rate of substitution $(E_tMRS_{\tilde{c}_{t+1},c_t})^{-1}$ yields $\beta(1+R_t^f)$. The n^{th} order Taylor approximation of $u'(\tilde{c}_{t+1})$ at c_t yields

$$\begin{aligned} E_tMRS_{\tilde{c}_{t+1},c_t} &= \frac{E_tu'(\tilde{c}_{t+1})}{u'(c_t)}, \\ &= \frac{1}{u'(c_t)}E_t(u'(c_t) + \sum_{i=1}^{n}\frac{c_t^i}{i!}u^{(i+1)}(c_t)(\tilde{g}_{t+1}-1)^i + o((\tilde{g}_{t+1}-1)^n)), \\ &= 1 + \sum_{i=1}^{n}\frac{(-1)^i}{i!}\,\mathrm{RR}_{i+1}(c_t)E_t[(\tilde{g}_{t+1}-1)^i] + o(g^n), \\ &= A(n) + o(g^n), \end{aligned} \tag{37}$$

where $\tilde{c}_{t+1} = c_t\tilde{g}_{t+1}$, $A(n) = 1 + \sum_{i=1}^{n} \frac{(-1)^i}{i!}\,\mathrm{RR}_{i+1}(c_t)E_t[(\tilde{g}_{t+1}-1)^i]$, $n \geq 1$, and $g = \max\{\bar{g}-1, 1-\underline{g}\}$. Thus, the commonly maintained assumption $E_t MRS_{\tilde{c}_{t+1},c_t} = 1$ implies that $A(n)-1$ is negligible. For instance, this assumption might hold for a very smooth consumption growth process.

A5.2. Standard Representation

The n^{th} order Taylor expansion of $u'(\tilde{c}_{t+1})$ at c_t in the standard representation of the C-CAPM in (1) gives

$$\begin{aligned}
\frac{E_t\tilde{R}_{t+1} - R_t^f}{\beta(1+R_t^f)} &= -\frac{1}{u'(c_t)}\,\mathrm{cov}_t(u'(c_t\tilde{g}_{t+1}), \tilde{R}_{t+1}),\\
&= -\frac{1}{u'(c_t)}\,\mathrm{cov}_t(u'(c_t) + \sum_{i=1}^{n}\frac{c_t^i}{i!}u^{(i+1)}(c_t)(\tilde{g}_{t+1}-1)^i + o((\tilde{g}_{t+1}-1)^n), \tilde{R}_{t+1}),\\
&= \sum_{i=1}^{n}\frac{(-1)^{i+1}}{i!}\,\mathrm{RR}_{i+1}(c_t)\,\mathrm{cov}_t((\tilde{g}_{t+1}-1)^i, \tilde{R}_{t+1}) + \mathrm{cov}_t(o((\tilde{g}_{t+1}-1)^n), \tilde{R}_{t+1}),\\
&= \sum_{i=1}^{n}\frac{(-1)^{i+1}}{i!}\,\mathrm{RR}_{i+1}(c_t)\,\mathrm{cov}_t((\tilde{g}_{t+1}-1)^i, \tilde{R}_{t+1}) + o(g^n). \qquad (38)
\end{aligned}$$

This approximation can be rewritten in terms of an absolute risk index $\mathrm{AR}_k(c) = (-1)^{k-1}u^{(k)}/u'$, by replacing $\mathrm{RR}_{i+1}(c)$ with $\mathrm{AR}_{i+1}(c)c^i$.

A5.3. FED Representation

The n^{th} order Taylor expansion of $u''(\tilde{c}_{t+1})$ at c_t in the FED representation of the C-CAPM in (3) gives

$$\begin{aligned}
\frac{E_t\tilde{R}_{t+1} - R_t^f}{\beta(1+R_t^f)} &= \int_{\underline{g}}^{\bar{g}} \mathrm{FED}(\tilde{R}_{t+1}|g_{t+1})F_{\tilde{g}_{t+1}}(g_{t+1})[-c_t\frac{u''(c_tg_{t+1})}{u'(c_t)}]dg_{t+1}\\
&= \int_{\underline{g}}^{\bar{g}} \mathrm{FED}(\tilde{R}_{t+1}|g_{t+1})F_{\tilde{g}_{t+1}}(g_{t+1})[-c_t\frac{u''(c_t) + \sum_{i=1}^{n}\frac{c_t^i}{i!}u^{(i+2)}(c_t)(g_{t+1}-1)^i + o((g_{t+1}-1)^n)}{u'(c_t)}]dg_{t+1}\\
&= \mathrm{RR}_2(c_t)\,\mathrm{cov}(\tilde{g}_{t+1}, \tilde{R}_{t+1}) + \sum_{i=1}^{n}(-1)^{i+2}\,\mathrm{RR}_{i+2}(c_t)\frac{1}{(i+1)!}\,\mathrm{cov}((\tilde{g}_{t+1}-1)^{i+1}, \tilde{R}_{t+1}) + o(g^{n+1})\\
&= \sum_{i=1}^{n+1}\frac{(-1)^{i+1}}{i!}\,\mathrm{RR}_{i+1}(c_t)\,\mathrm{cov}((\tilde{g}_{t+1}-1)^i, \tilde{R}_{t+1}) + o(g^{n+1}). \qquad (39)
\end{aligned}$$

It is straightforward to restate the above approximation in terms of an absolute risk index $\mathrm{AR}_k(c) = (-1)^{k-1}u^{(k)}/u'$, by using the equality $\mathrm{RR}_{i+1}(c) = \mathrm{AR}_{i+1}(c)c^i$.

A5.4. SED Representation

The n^{th} order Taylor expansion of $u''(\tilde{c}_{t+1})$ and $u'''(\tilde{c}_{t+1})$ at c_t in the SED representation of the C-CAPM in (5) gives

$$\begin{aligned}
\frac{E_t\tilde{R}_{t+1} - R_t^f}{\beta(1+R_t^f)} &= \mathrm{cov}_t(\tilde{g}_{t+1}, \tilde{R}_{t+1})(-c_t\frac{u''(c_t\bar{g})}{u'(c_t)}) + \int_{\underline{g}}^{\bar{g}} \mathrm{SED}(\tilde{R}_{t+1}|g_{t+1})(-c_t^2\frac{u'''(c_t g_{t+1})}{u'(c_t)})dg_{t+1} \\
&= \mathrm{cov}_t(\tilde{g}_{t+1}, \tilde{R}_{t+1})(-c_t\frac{u''(c_t) + \sum_{i=1}^n \frac{c_t^i}{i!}u^{(i+2)}(c_t)(\bar{g}-1)^i + o((\bar{g}-1)^n)}{u'(c_t)}) \\
&\quad + \int_{\underline{g}}^{\bar{g}} \mathrm{SED}(\tilde{R}_{t+1}|g_{t+1})(-c_t^2\frac{u'''(c_t) + \sum_{i=1}^n \frac{c_t^i}{i!}u^{(i+3)}(c_t)(g_{t+1}-1)^i + o((g_{t+1}-1)^n)}{u'(c_t)})dg_{t+1} \\
&= \mathrm{cov}_t(\tilde{g}_{t+1}, \tilde{R}_{t+1})[\mathrm{RR}_2(c_t) + \sum_{i=1}^n \frac{(-1)^{i+2}}{i!}\mathrm{RR}_{i+2}(c_t)(\bar{g}-1)^i] \\
&\quad + \int_{\underline{g}}^{\bar{g}} \mathrm{SED}(\tilde{R}_{t+1}|g_{t+1})[-\mathrm{RR}_3(c_t) + \sum_{i=1}^n \frac{(-1)^{i+3}}{i!}\mathrm{RR}_{i+3}(c_t)(g_{t+1}-1)^i]dg_{t+1} + o(g^{n+2}) \\
&= \mathrm{cov}_t(\tilde{g}_{t+1}, \tilde{R}_{t+1})[\sum_{i=1}^{n+1} \frac{(-1)^{i+1}}{(i-1)!}\mathrm{RR}_{i+1}(c_t)(\bar{g}-1)^{i-1}] \\
&\quad + \int_{\underline{g}}^{\bar{g}} \mathrm{SED}(\tilde{R}_{t+1}|g_{t+1})[\sum_{i=1}^{n+1} \frac{(-1)^{i+2}}{(i-1)!}\mathrm{RR}_{i+2}(c_t)(g_{t+1}-1)^{i-1}]dg_{t+1} + o(g^{n+2}),
\end{aligned} \tag{40}$$

where $\mathrm{SED}(\tilde{R}_{t+1}|g_{t+1}) = \mathrm{cov}_t(-(g_{t+1} - \tilde{g}_{t+1})_+, \tilde{R}_{t+1})$ from (6). One can use $\mathrm{RR}_{i+1}(c) = \mathrm{AR}_{i+1}(c)c^i$ to rewrite this expression in terms of an absolute risk index $\mathrm{AR}_k(c) = (-1)^{k-1}u^{(k)}/u'$.

Empirically, one can assess the integrated SED quantity as

$$\int_{\underline{g}}^{\bar{g}} \mathrm{SED}(\tilde{R}|g)dg \approx \sum_{l=1}^{Q} \mathrm{cov}_t(-(g^{(l)} - \tilde{g})_+, \tilde{R}) \times (g^{(l)} - g^{(l-1)}) \tag{41}$$

where $\underline{g} = g^{(0)} \leq \cdots \leq g^{(l)} \leq \cdots \leq g^{(Q)} = \bar{g}$ are quantiles values that partition the consumption growth sample into Q subsets of equal sizes. We set $Q = N_{obs}/2$ for our calculations.

A5.5. Generalization to k^{th}-Degree Expectation Dependence C-CAPM

Recall that there is a recursive relationship between consecutive orders of expectation dependence, that is, $K^{th}ED(\tilde{R}_{t+1}|g_{t+1}) = \int_{\underline{g}}^{g_{t+1}}(K-1)^{th}ED(\tilde{R}_{t+1}|s)ds$. Using iterated integrations by parts, one can generalize the theoretical results of Sections 3 and 4 to higher-degree expectation dependence risks – 3^{rd}-degree, 4^{th}-degree, ..., k^{th}-degree – and corresponding higher-order risk attitudes – 4^{th}-order or temperance, 5^{th}-order or edginess, ..., $(k+1)^{th}$-order – of representative agents, with $k \geq 3$.

A $K^{th}ED$ representation of the C-CAPM pricing relation, for $K \geq 3$, writes

$$\begin{aligned}\frac{E_t\tilde{R}_{t+1} - R_t^f}{\beta(1+R_t^f)} &= \sum_{k=2}^{K} k^{th}ED(\tilde{R}_{t+1}|\overline{g})[(-1)^{k-1}c_t^{k-1}\frac{u^{(k)}(c_t\overline{g})}{u'(c_t)}] \\ &+ \int_{\underline{g}}^{\overline{g}} K^{th}ED(\tilde{R}_{t+1}|g_{t+1})[(-1)^K c_t^K \frac{u^{(K+1)}(c_t g_{t+1})}{u'(c_t)}]dg_{t+1}. \end{aligned} \qquad (42)$$

A6. The Hansen's (1994) skew t distribution

The likelihood function for the skew t distribution proposed by Hansen (1994) is:

$$f(z|\nu,\lambda) = \begin{cases} bc\left(1+\frac{1}{\nu-2}\left(\frac{bz+a}{1-\lambda}\right)^2\right)^{-(\nu+1)/2}, & if\ z < -a/b, \\ bc\left(1+\frac{1}{\nu-2}\left(\frac{bz+a}{1+\lambda}\right)^2\right)^{-(\nu+1)/2}, & if\ z \geq -a/b, \end{cases}$$

where the degree of freedom or fat-tailedness parameter is bounded as $2 < \nu < \infty$, and the skewness parameter is bounded as $-1 < \lambda < 1$. The other constants are given by:

$$a = 4\lambda c\left(\frac{\nu-2}{\nu-1}\right),$$

$$b^2 = 1 + 3\lambda^2 - a^2,$$

and

$$c = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi\left(\nu-2\right)}\Gamma\left(\frac{\nu}{2}\right)}.$$

Hansen (1994) shows that the mean, mode, and variance of this density function are $Ez = 0$, $Mz = -a/b$, and $Vz = 1$. When $\lambda = 0$, the distribution reduces to the (standard) symmetric Sudent's t distribution. When $\lambda < 0$ or $\lambda > 0$, the distribution is skewed to the left or right, respectively. Setting $\nu \to \infty$ (in practice to a number higher than 30) and $\lambda = 0$ yields a standard normal distribution.

Now consider the transformed random variable $y^* = bz + a$. Let $\mu_n \equiv E\left[(y^* - Ey^*)^n\right]$ and $m_n \equiv E\left[y^{*n}\right]$ denote the n^{th} central and noncentral moments. It is straightforward to see that the mean is $m_1 = a$ and the variance is $\mu_2 = m_2 - m_1^2 = b^2$. The skewness is

$$Sy^* = Sz \equiv \frac{\mu_3}{\mu_2^{3/2}} = \frac{m_3 - 3m_1m_2 + 2m_1^3}{b^3},$$

and kurtosis is

$$Ky^* = Kz \equiv \frac{\mu_4}{\mu_2^2} = \frac{m_4 - 4m_1m_3 + 6m_1^2m_2 - 3m_1^4}{b^4},$$

where

$$\begin{aligned}
m_1 &= 4\lambda c\left(\frac{\nu-2}{\nu-1}\right) = a, \\
m_2 &= 1 + 3\lambda^2 = b^2 + a^2, \\
m_3 &= 16c\lambda\left(1+\lambda^2\right)\frac{\left(\nu-2\right)^2}{\left(\nu-1\right)\left(\nu-3\right)},\ if\ \nu > 3, \\
m_4 &= 3\left(1 + 10\lambda^2 + 5\lambda^4\right)\frac{\left(\nu-2\right)}{\left(\nu-4\right)},\ if\ \nu > 4.
\end{aligned}$$

For an AR(1) process $y_{t+1} = Ey + \phi_y\left(y_t - Ey\right) + \sqrt{\left(1-\phi_y^2\right)Vy}\, z_{t+1}$, with skew t innovations z_{t+1}, unconditional mean Ey, and variance Vy as in (13), the skewness is given by

$$Sy = Sz\frac{\left(1-\phi^2\right)^{3/2}}{1-\phi^3},$$

and kurtosis is

$$Ky = \frac{Kz\,(1-\phi^2)+6\phi^2}{1+\phi^2}.$$

When ϕ tends to 1, the unconditional skewness Sy tends to 0 and the unconditional kurtosis Ky tends to 3 (no excess kurotsis). When ϕ tends to 0, Sy and Ky tends to the skewness and kurtosis of the skew t innovation z_{t+1}.

A7. Details on the Construction of Variance Risk Premium

A7.1. Building Physical Variances from High Frequency Returns

To empirically build the realized variance series, we employ high frequency observations on S&P 500 index. On a given day, we use the last record in each five-minute interval to construct a grid of five-minute equity index log-returns. As in Andersen et al. (2003), the realized variance on each transaction day t is given by $\tilde{V}_t = \sum_{j=1}^{n_t} \tilde{r}_{j,t}^2$, where $\tilde{r}_{j,t}^2$ is the j^{th} intraday squared log-return and n_t denotes the number of intraday records. Note that we observe $n_t = 78$ five-minute returns on a typical transaction day. We add the squared overnight log-return (the close-to-open change in log price) and apply the standard scaling, thus ensuring that the sample average of the realized variance matches the sample variance of daily log-returns. The cumulative realized variance for a given horizon τ is given by the aggregation daily realized variances, $\tilde{V}_{t,\tau} = \sum_{d=1}^{\tau} \tilde{V}_{t+d}$. In our empirical analysis we consider $\tau = 1$-month horizon to get monthly series.

To get genuine conditional expectation of the realized variance under the physical probability measure (P), we use Corsi's (2009) heteroscedastic autoregressive realized variance (HAR-RV) specification. Even though various econometric frameworks exist, the HAR-RV model allows to generate reliable forecasts. Alternatively, on may use a random walk model as in Bollerslev et al. (2009). The HAR-RV model, which features multi-frequency (daily, weekly, monthly) predictors, is cast as

$$\tilde{V}_{t+1} = E_t^P[\tilde{V}_{t+1}] + \tilde{\zeta}_{t+1}, \tag{43}$$

where $E_t^P[\tilde{V}_{t+1}] = \alpha_0 + \alpha_d V_t + \alpha_w V_{t,w} + \alpha_m V_{t,m}$, $w = 5$ $days$ (resp. $m = 20$ $days$) denotes one week (resp. one month) trading period, and $\tilde{\zeta}_{t+1}$ is an innovation term.

A7.2. Extracting Risk-Neutral Variances from Options

To extract model-free variance series under the risk-neutral probability measure (Q), we use daily European options written on the S&P 500 index. Our option panel from OptionMetrics runs from September 03, 1996 to August 31, 2015. Call and put contracts are sorted by maturity and strike price. We use mid-quotes and apply the standard filters as in Chang et al. (2013). Namely, we discard options with zero transaction volume, options with mid-quotes less than \$3/8, and options which do not satisfy basic no-arbitrage bounds.

It is important to mention that actual option data are available for a discontinuous and limited set of strike prices whereas the construction of risk-neutral moments requires to compute integrals (or weighted portfolios of) option contracts over a compact set of strike prices. Thus, on each day and for any given maturity, we map option prices into Black and Scholes' (1973) implied volatilities. We employ a cubic spline to interpolate the observed implied volatilities over a finely-discretized moneyness grid and generate a continuum of implied volatilities. The interpolation is performed only for dates where at least two OTM call prices and two OTM put prices are available. In addition, we extrapolate the implied volatility of the lowest or highest available strike price outside (below or above) the observed moneyness range of any given contract, as implemented in Chang et al. (2013). We then map these interpolated-extrapolated implied volatilities back into call and put prices using the Black and Scholes (1973) equation.

The squared VIX can be used to proxy the one-month ahead risk-neutral variance as in Bollerslev et al. (2009), though it might induce some biases (Carr et al. 2012). To build risk-neutral variance series for a generic horizon τ, we rely on the nonparametric methodology implemented in Bakshi et al. (2003). A key insight of this approach is that one can replicate any desired payoff by designing a portfolio of OTM European call and put options over a continuum of strike prices. To outline this approach, let $\tilde{r}_{t,\tau} = log(\tilde{x}_{t+\tau}) - log(\tilde{x}_t)$ denote the log-return on the underlying asset between time t and $t+\tau$. Moreover, consider the payoff at maturity of a contingent claim $\mathcal{A}[\tilde{x}] = \tilde{r}^n_{t,\tau}$ that describes a power ($n = 2$-quadratic, 3-cubic, 4-quartic, etc.) contract. As shown in Bakshi and Madan (2000), any twice-continuously differentiable payoff with bounded expectation can be spanned as

$$\begin{aligned} \mathcal{A}[\tilde{x}] &= \mathcal{A}[\overline{x}] + (\tilde{x} - \overline{x})\mathcal{A}_x[\overline{x}] + \int_{\overline{x}}^{\infty} \mathcal{A}_{xx}[K](\tilde{x} - K)^+ dK \\ &\quad + \int_0^{\overline{x}} \mathcal{A}_{xx}[K](K - \tilde{x})^+ dK. \end{aligned} \tag{44}$$

This spanning expression entails positions in the slope (first derivative $\mathcal{A}_x[\bullet]$ evaluated at some $\overline{x}$) and the curvature (second derivative $\mathcal{A}_{xx}[\bullet]$ evaluated at the strike price K) of the payoff function. By discounting the risk-neutral conditional expectation of the contingent claim at the log risk-free rate $r^f = log(1 + R^f)$, we obtain its price

$$\begin{aligned} E_t^Q\{e^{-r^f\tau}\mathcal{A}[\tilde{x}]\} &= e^{-r^f\tau}(\mathcal{A}[\overline{x}] - \overline{x}\mathcal{A}_x[\overline{x}]) + \mathcal{A}_x[\overline{x}]x_t + \int_{\overline{x}}^{\infty} \mathcal{A}_{xx}[K]C(t,\tau;K)dK \\ &\quad + \int_0^{\overline{x}} \mathcal{A}_{xx}[K]P(t,\tau;K)dK, \end{aligned} \tag{45}$$

which corresponds to a portfolio including the risk-free bond, the underlying asset, and OTM calls and puts. We implement numerical approximations of the integrals in (45). The risk-neutral variance ($n = 2$) is computed as

$$V^Q_{t,\tau} = E_t^Q[(\tilde{r}_{t,\tau} - E_t^Q[\tilde{r}_{t,\tau}])^2]. \tag{46}$$